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VECTOR CALCULUS 725



Distance formula in three dimensions

The distance $|P_1 P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Linear algebra:

$$|P_1 P_2| = \sqrt{\overrightarrow{P_1 P_2} \cdot \overrightarrow{P_1 P_2}}$$

normal

E1 Determine whether the three points $P = (-6, 2, 10)$, $Q = (-9, -4, 1)$, $R = (-12, -10, -8)$ are collinear by computing the distance between pairs of points.

Distance from P to Q :

$$|PQ| = \sqrt{(-9 - (-6))^2 + (-4 - 2)^2 + (1 - 10)^2} = \sqrt{(-3)^2 + (-6)^2 + (-9)^2} = \sqrt{9 + 36 + 81} = \sqrt{126}$$

Linear algebra way:

$$\overrightarrow{PQ} = [(-9 - (-6)), -4 - 2, 1 - 10] = [-3, -6, -9]$$

$$|PQ| = \sqrt{\overrightarrow{PQ} \cdot \overrightarrow{PQ}} = \sqrt{9 + 36 + 81} = \sqrt{126}$$

Distance from Q to R :

$$\overrightarrow{QR} = [-3, -6, -9]$$

$$|QR| = \sqrt{\overrightarrow{QR} \cdot \overrightarrow{QR}} = \sqrt{9 + 36 + 81} = \sqrt{126}$$

Distance from P to R :

$$\overrightarrow{PR} = [-6, -12, -18]$$

$$|PR| = \sqrt{36 + 144 + 324} = \sqrt{504}$$

Actually, they are the same thing

∴ collinear

Equation of a Sphere

An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

In particular, if the center is the origin O , then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$

P5. Find the equation of the sphere centered at $(-6, -3, -10)$ with radius 8.

$$(x - (-6))^2 + (y - (-3))^2 + (z - (-10))^2 = 8^2$$

intersection 题型

$$(x+6)^2 + (y+3)^2 + (z+10)^2 = 64$$

Give an equation which describes the intersection of this sphere with the plane $z = -9$.

∴ The equation of the sphere is $(x+6)^2 + (y+3)^2 + (z+10)^2 = 64$

Plug $z = -9$ into above equation:

$$(x+6)^2 + (y+3)^2 = 63$$



P6. Find the equation of the sphere if one of its diameters has endpoints $(10, 1, -4)$ and $(12, 5, 2)$. 已知直径, 求球方程

$$\text{let } A = (10, 1, -4) \quad B = (12, 5, 2)$$

$$\vec{AB} = (2, 4, 6)$$

$$|\vec{AB}| = \sqrt{\vec{AB} \cdot \vec{AB}} = \sqrt{4+16+36} = \sqrt{56} \quad (\text{diameters of sphere})$$

$$\therefore r = \frac{|\vec{AB}|}{2} = \frac{\sqrt{56}}{2}$$

$$c = A + r = (10, 1, -4) + \left(\frac{124, 61}{2} \right) = (10, 1, -4) + (1, 2, 3) = (11, 3, -1)$$

key

$$\therefore (x-1)^2 + (y-3)^2 + (z+1)^2 = \left(\frac{\sqrt{56}}{2}\right)^2 = 14$$

P7. Find an equation of the sphere that passes through the origin and whose center is $(2, -10, -3)$.

Let $A = (2, -10, -3)$ $B = (0, 0, 0)$ 已知半径

$$\overline{BA} = (2, -10, -3)$$

$$\therefore (x-2)^2 + (y+10)^2 + (z+3)^2 = \sqrt{13}^2 = 113$$

Ps. Find an equation of the largest sphere with center $(10, 9, 5)$ that is contained completely in the first octant. 第一卦限

$$\text{in the first octant, } (x-10)^2 + (y-9)^2 + (z-5)^2 - 25 = 0$$

Section 10.3 P10 Webwork

A constant force $\mathbf{F} = -1\mathbf{i} + 2\mathbf{j} + 1\mathbf{k}$ moves an object along a straight line from the point $(-3, 0, 0)$ to the point $(2, 2, -2)$. Find the work done if the distance is measured in meters and the force in newtons.

$$W = |\mathbf{F}| \cdot |\mathbf{D}| \cdot \cos \theta = \vec{\mathbf{F}} \cdot \vec{\mathbf{D}}$$

The work done by a constant force \mathbf{F} is the dot product $\vec{\mathbf{F}} \cdot \vec{\mathbf{D}}$, where $\vec{\mathbf{D}}$ is the displacement vector.

$$\therefore W = [-1, 2, 1] \cdot [2 - (-3), 2 - 0, -2 - 0] = [-1, 2, 1] \cdot \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} = -5 + 4 - 2 = -3 \text{ J}$$

功可是负数

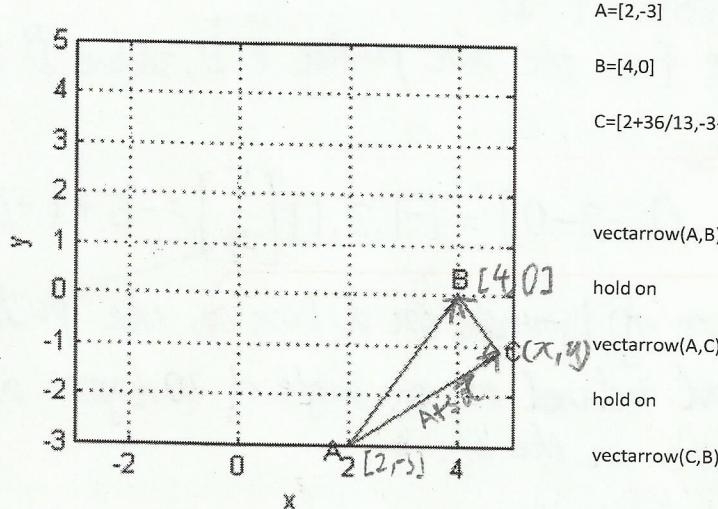
- P11 A woman exerts a horizontal force of 1 pounds on a box as she pushes it up a ramp that is 1 feet long and inclined at an angle of 30 degrees above the horizontal. Find the work done on the box



$$W = |\mathbf{F}| \cdot |\mathbf{S}| \cos \theta = |X| \times \cos\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) \text{ ft-lb}$$

key!

Gandalf the Grey started in the Forest of Mirkwood at a point with coordinates $(2, -3)$ and arrived in the Iron Hills at the point with coordinates $(4, 0)$. If he began walking in the direction of the vector $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$ and changes direction only once, when he turns at a right angle, what are the coordinates of the point where he makes the turn.



Line equation
See University of Toronto Linear algebra notes.

$$\vec{AC} = \vec{A} + s\vec{d} = [2, -3] + s[3, 2] = [2+3s, -3+2s]$$

Let C be $[x, y]$, and C is a point on line \vec{AC} .

$$\therefore x = 2+3s \quad (1)$$

$$y = -3+2s \quad (2)$$

And $\because \vec{AC} \perp \vec{CB}$

$$\therefore \vec{d} \cdot \vec{CB} = 0 \quad (\vec{AC} \text{ and } \vec{d} \in V, \vec{CB} \in V^\perp)$$

$$[3, 2] \cdot [4-x, 0-y] = 0 \Rightarrow 12-3x-2y = 0 \Rightarrow y = \frac{12-3x}{2} \quad (3)$$

According to (1) (2) (3) we could obtain:

$$\begin{cases} \frac{12-3x}{2} = -3+2s \\ x = 2+3s \end{cases} \Rightarrow s = \frac{12}{13} \quad x = 2 + \frac{36}{13}$$

Plug $s = \frac{12}{13}$ into (2), we could get

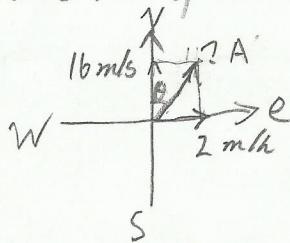
$$y = -3 + \frac{24}{13}$$

\therefore The point C is $[2 + \frac{36}{13}, -3 + \frac{24}{13}]$



3. A child walks due east on the deck of a ship at 2 miles per hour. The ship is moving north at a speed of 16 miles per hour.

Find the speed and direction of the child relative to the surface of the water.



$$A = \sqrt{16^2 + 2^2} = 16.125 \text{ mph}$$

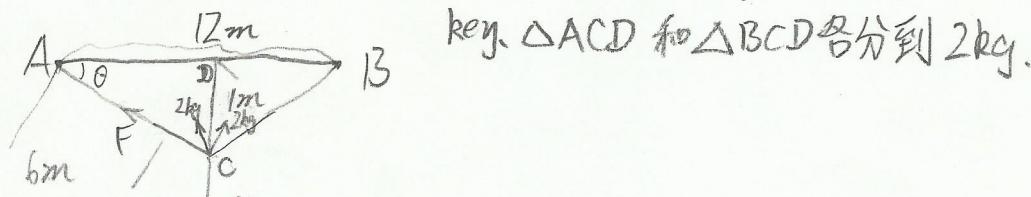
The angle of the direction from the north =

$$\sin \theta = \frac{2}{16.125} \Rightarrow \theta = \sin^{-1} \left(\frac{2}{16.125} \right) = 0.12435 \text{ radians}$$

4. A horizontal clothesline is tied between 2 poles, 12 meters apart. When a mass of 4 kilograms is tied to the middle of the clothesline, it sags a distance of 1 meter.

What is the magnitude of the tension on the ends of the clothesline?

Tension = ?



$$\tan \theta = \frac{1}{6} \Rightarrow \theta = 9.462322^\circ$$

$$\sin \theta = \frac{2 \text{ kg} \cdot 9.8}{F} \Rightarrow F = 119.22 \text{ N}$$

Sec 10.3. 6.

What is the angle in radians between the vectors $a = [6, -8, 8]$ and $b = [2, 8, 5]$?

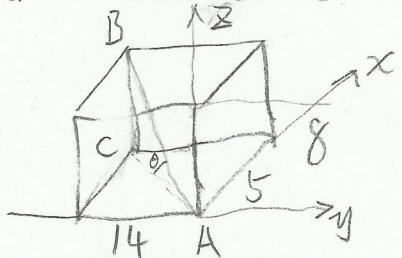
$$A \cdot B = |A| \cdot |B| \cos \theta \Rightarrow \cos \theta = \frac{A \cdot B}{|A| \cdot |B|} = \frac{12 + (-64) + 40}{\sqrt{36+64+64} \cdot \sqrt{4+64+25}}$$

$$\theta = \cos^{-1} \left(\frac{-12}{\sqrt{164} \cdot \sqrt{93}} \right) = 1.66811 \text{ radians}$$





8. A rectangular box has length 14 inches, width 5 inch and height of 8 inches. Find the angle between the diagonal of the box and the diagonal of its base. The angle should be measured in radians.



$$A[0, 0, 0] \quad B[5, 0, 8] \quad C[14, 0, 0]$$

$$\vec{AC} [5, -14, 0] \quad \vec{AB} [5, 0, 8]$$

$$\cos \theta = \frac{\vec{AC} \cdot \vec{AB}}{|\vec{AC}| \cdot |\vec{AB}|} = \frac{25 + 196 + 0}{\sqrt{25+196} \cdot \sqrt{25+196+64}} = \frac{221}{\sqrt{221} \cdot \sqrt{285}} \Rightarrow \theta = 0.4936 \text{ radians}$$

Page 7.

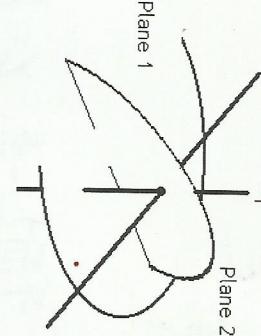
The Line of Intersection between two Planes

Problem:

Find the line of intersection between the two planes given by the vector equations

$$\mathbf{r}_1 \cdot [3, 4, 0] = 5 \text{ and } \mathbf{r}_2 \cdot [1, 2, 3] = 6$$

A diagram of this is shown on the right.



\mathbf{n}_1 is a normal vector to Plane 1.
 \mathbf{n}_2 is a normal vector to Plane 2.
 By simple geometrical reasoning, the line of intersection is perpendicular to both normals.

To find the position vector, \mathbf{r} , of any point on the line of intersection;

find a vector, \mathbf{v} , to which the line is parallel,

find the position vector, \mathbf{a} , of specific point on the line,

then; $\mathbf{r} = \mathbf{a} + t\mathbf{v}$, is the required result.

therefore $\mathbf{n}_1 \times \mathbf{n}_2 = [3, 4, 0] \times [1, 2, 3]$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 0 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= [4, 0] \mathbf{i} - [3, 0] \mathbf{j} + [1, 3] \mathbf{k}$$

$$= [12, -9, 9]$$

1.

To find \mathbf{a} , 隨便找個 滿足兩個Planes 方程

at any point on the line the position vector, $\mathbf{r}_1 = \mathbf{r}_2$
 Let the point be (x, y, z) , then;

From the equations to the planes,

$$(x, y, z) \cdot [3, 4, 0] = 5 \text{ and } (x, y, z) \cdot [1, 2, 3] = 6$$

$$3x + 4y = 5 \text{ and } x + 2y + 3z = 6$$

Let the point have an x co-ordinate = 0 then;

$$y = \frac{5}{4} \text{ and } z = \frac{7}{6}$$

$$\mathbf{a} \text{ is therefore } [0, \frac{5}{4}, \frac{7}{6}],$$

Substituting result 1. and 2. into $\mathbf{r} = \mathbf{a} + t\mathbf{v}$ gives;

$$\mathbf{r} = [0, \frac{5}{4}, \frac{7}{6}] + t[12, -9, 9]$$

or, more simply, $\mathbf{r} = [0, \frac{5}{4}, \frac{7}{6}] + t[4, -3, 3]$

since the vector $[12, -9, 9]$ is parallel to $[4, -3, 3]$

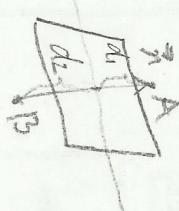
(717, #51) Find an equation for the plane consisting of all points that are equidistant from the points $(1, 1, 0)$ and $(0, 1, 1)$. means $d_1 = d_2$

Solution: The midpoint between the two points will be a point on the plane. This point is $\left(\frac{1+0}{2}, \frac{1+1}{2}, \frac{0+1}{2}\right) = \left(\frac{1}{2}, 1, \frac{1}{2}\right)$

$$\frac{\overrightarrow{AB}}{2} = \overrightarrow{A + B}$$

The direction vector between the two points, which is also a normal to the plane, is:

$$\mathbf{n} = <0 - 1, 1 - 1, 1 - 0> = <-1, 0, 1>$$



The general equation for a plane is:

$$\mathbf{n} \bullet <x - x_0, y - y_0, z - z_0> = 0$$

$$<-1, 0, 1> \bullet <x - \frac{1}{2}, y - 1, z - \frac{1}{2}> = 0$$

平面方程

$$\mathbf{n} \cdot [\mathbf{r} - \mathbf{r}_0] = [a, b, c] \cdot [x - x_0, y - y_0, z - z_0]$$

$$-1(x - \frac{1}{2}) + 0 + 1(z - \frac{1}{2}) = 0$$

$$-x + \frac{1}{2} + z - \frac{1}{2} = 0$$

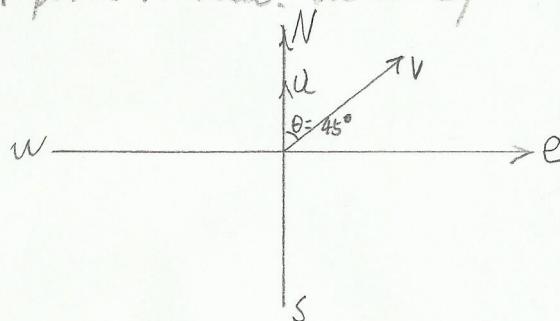
$$x = z$$

by: $\mathbf{n} \parallel$

P3. Web work

You are looking down at a map. A vector u with $|u|=2$ points north and a vector v with $|v|=7$ points northeast. The cross product $u \times v$ points: down

右定则



key $|u \times v| = |u| \cdot |v| \sin \theta$

The magnitude $|u \times v| = 2 \times 7 \times \sin 45^\circ = 14 \times \frac{\sqrt{2}}{2} = 7\sqrt{2}$

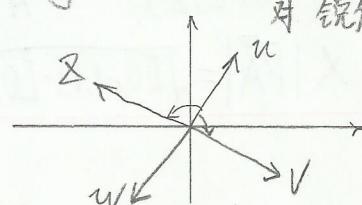
P4 webwork

Think of the letter X as four vectors starting from the center and pointing outward. Label the four vectors starting from the top left and proceeding clockwise as u, v, w, z .

Does $u \times v$ point in or out of the page? in

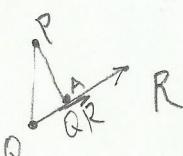
Does $u \times z$ point in or out of the page? out

Compute $u \times w = [0, 0, 0]$? why?



P5 webwork

Find the distance the point $P(1, -4, -6)$ is to the line through the two points $Q(5, -1, -5)$ and $R(3, -4, -4)$



$$\vec{QR} = [-2, -3, 1] \quad \vec{QP} = [-4, -3, -1] \quad |\vec{QP}| = \sqrt{\vec{QP} \cdot \vec{QP}} = \sqrt{16+9+1} = \sqrt{26}$$

$$\text{key } \vec{QA} = \text{Proj}_{QR} \vec{QP} = \frac{\vec{QP} \cdot \vec{QR}}{|\vec{QR}|^2} \cdot \vec{QR} = \frac{-8+9-1}{4+9+1} \cdot [-2, -3, 1] = \frac{8}{7} \cdot [-2, -3, 1] = [-\frac{16}{7}, -\frac{24}{7}, \frac{8}{7}]$$

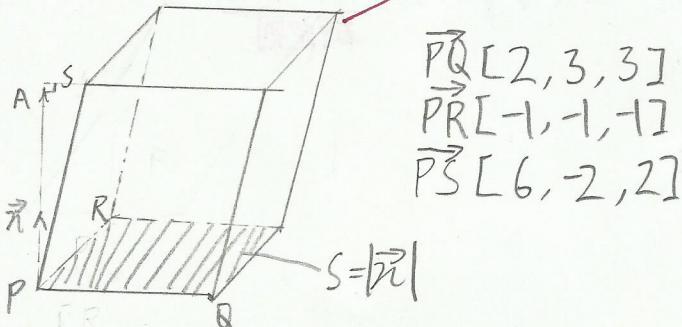
$$\text{Proj}_a b = \frac{a \cdot b}{|a|^2} a \quad [b \text{ 在 } a \text{ 上的投影}]$$

$$|\vec{QA}| = \frac{8}{7} \cdot \sqrt{4+9+1} = \frac{8}{7} \times \sqrt{14} = 4.2762$$

$$\therefore |\vec{PA}|^2 = 26 - \frac{64}{49} \times 14 = 7.71428$$

$$\therefore |\vec{PA}| = 2.7774$$

P. Find the volume of the parallelepiped with adjacent edges \vec{PQ} , \vec{PR} , \vec{PS} , where
 $P[1, 2, 1]$ $Q[3, 5, 4]$ $R[0, 1, 0]$ $S[7, 0, 3]$



$$\begin{aligned}\vec{PQ} &= [2, 3, 3] \\ \vec{PR} &= [-1, -1, -1] \\ \vec{PS} &= [6, -2, 2]\end{aligned}$$

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 2 & 3 & 3 \\ -1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 3 \\ -1 & -1 \end{vmatrix} i + \begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} j + \begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} k = 0i - 1j + 1k = [0, -1, 1]$$

$$\vec{PA} = \text{Proj}_{\vec{n}} \vec{PS} = \frac{\vec{PS} \cdot \vec{n}}{|\vec{n}|^2} \cdot \vec{n} = \frac{[6, -2, 2] \cdot [0, -1, 1]}{2} \cdot [0, -1, 1] = 2 \cdot [0, -1, 1]$$

$$\therefore |\vec{PA}| = \sqrt{[0, -2, 2] \cdot [0, -2, 2]'} = \sqrt{8}$$

$$\therefore \text{Volume} = |\vec{n}| \times |\vec{PA}| = \sqrt{[0, -1, 1] \cdot [0, -1, 1]'} \times \sqrt{8} = \sqrt{2} \times \sqrt{8} = \sqrt{16} = 4$$

Consider the line which passes through the point $P(-2, -4, 1)$, and which is parallel to the line $x=1+3t, y=2+3t, z=3+7t$. Find the point of intersection of this new line with each of the coordinate planes: xy -plane, xz -plane, yz -plane.

$$\begin{aligned} x &= 1+3t & y &= 2+3t & z &= 3+7t \end{aligned} \sim [1, 2, 3] + s[3, 3, 7] \vec{d}$$

\therefore the line passes through the point $P(-2, -4, 1)$

The equation of this line is $[-2, -4, 1] + s[3, 3, 7] \sim x = -2 + 3s, y = -4 + 3s, z = 1 + 7s$

xy -plane: $\Rightarrow z=0$ key 計算 s

$$\therefore z = 1 + 7s \Rightarrow 0 = 1 + 7s \Rightarrow s = -\frac{1}{7}$$

$$\therefore x = -2 + 3s = -2 + 3 \times (-\frac{1}{7}) = -2 - \frac{3}{7} = -\frac{17}{7}$$

$$y = -4 + 3s = -4 + 3 \times (-\frac{1}{7}) = -4 - \frac{3}{7} = -\frac{31}{7}$$

$$\therefore [-\frac{17}{7}, -\frac{31}{7}, 0]$$

xz -plane: $\Rightarrow y=0$

$$\therefore y = -4 + 3s \Rightarrow 0 = -4 + 3s \Rightarrow s = \frac{4}{3}$$

$$\therefore x = -2 + 3s = -2 + 3 \times \frac{4}{3} = -2 + 4 = 2$$

$$z = 1 + 7s = 1 + 7 \times \frac{4}{3} = \frac{3}{3} + \frac{28}{3} = \frac{31}{3}$$

$$\therefore [2, 0, \frac{31}{3}]$$

yz -plane: $\Rightarrow x=0$

$$\therefore x = -2 + 3s \Rightarrow 0 = -2 + 3s \Rightarrow s = \frac{2}{3}$$

$$y = -4 + 3s = -4 + 3 \times \frac{2}{3} = -2$$

$$z = 1 + 7s = 1 + 7 \times \frac{2}{3} = \frac{3}{3} + \frac{14}{3} = \frac{17}{3}$$

$$\therefore [0, -2, \frac{17}{3}]$$

Q8 Find the vector equation for the line of intersection of the planes $4x+4y+3z=-2$ and $4x+4z=-1$

$$r = [?, ?, 0] + t[16, ?, ?]$$

重点题型, 看Page 7, 求两面交线方程

n_1, n_2 of two planes are $[4, 4, 3], [4, 0, 4]$

$$\vec{n}_1 \times \vec{n}_2 = \begin{bmatrix} i & j & k \\ 4 & 4 & 3 \\ 4 & 0 & 4 \end{bmatrix} = [4 \cdot 4]i - [4 \cdot 4]j + [4 \cdot 0]k = 16i - 4j - 16k$$

Let A be a point at Intersection. Let $z=0$ (这里题目指定 $z=0$, 不指定可随便取, 在Intersection上就行)

$$\text{plane 1 - plane 2: } \Rightarrow 4y - z = -1 \quad \because z=0 \Rightarrow 4y - 0 = -1 \Rightarrow y = -\frac{1}{4}$$

$$\text{According to plane 2: } 4x + 4z = -1 \quad \because z=0 \Rightarrow 4x + 0 = -1 \Rightarrow x = -\frac{1}{4}$$

$$\therefore A = [-\frac{1}{4}, -\frac{1}{4}, 0] \quad \vec{d} = [16, -4, -16]$$

Pq Consider the two lines 題型，已知兩線參數方程，求交點

$$L_1: x_1 = -2t, y_1 = 1+2t, z_1 = 3t \text{ and}$$

$$L_2: x_2 = -7+3s, y_2 = 1+4s, z_2 = 4+2s$$

Find the point of intersection of the two lines.

判斷兩線是否相交，看它們的向量是否 linear dependence
 這題 $\vec{v}_1 = [-2, 2, 3], \vec{v}_2 = [3, 4, 2]$, 沒有 s 可使
 $[-2, 2, 3] = s[3, 4, 2] \Rightarrow$ 線性无关 \Rightarrow 相交
 不平行(平面)

$$\text{Let } x_1 = x_2, y_1 = y_2$$

$$\begin{cases} -7+3s = -2t \\ 1+2t = 1+4s \end{cases} \quad (x_1 = x_2) \Rightarrow \begin{cases} -7+3s + 2t = 0 \\ 1+2t = 1+4s \end{cases} \Rightarrow \begin{cases} -7+3s + 2t = 0 \\ 2t - 4s = 0 \end{cases} \Rightarrow 7s = 7 \Rightarrow s = 1$$

Plug $s=1$ into L_2 :

$$x_2 = -4, y_2 = 5, z_2 = 6$$

$$\therefore [-4, 5, 6]$$

Section 10.5: Problem 11

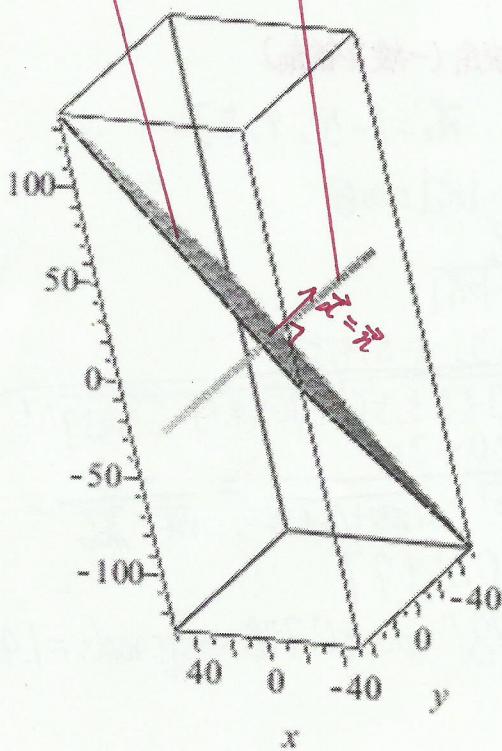
(1 pt)

Find an equation of a plane through the point $(-2, 0, 2)$ which is orthogonal to the line

$$x = 1 + 4t, y = -3 + 4t, z = -2 - 3t$$

in which the coefficient of x is 4.

$$\boxed{4x+4y-3z+14} = 0.$$

Question Type: 找出垂直于直线的面。

Key: 如果面垂直于直线，那么由于面的 \vec{n} 垂直于面，所以面的 \vec{n} 和直线的 \vec{d} 重合（如左图）。

$$\therefore \vec{d} = \vec{n} = [4, 4, -3] \quad P=[-2, 0, 2]$$

\therefore 这道题变成了已知 P 和 \vec{n} 求平面方程。

According to Plane equation formula

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \quad \vec{n} = [a, b, c] \quad \vec{r}_0 = [x_0, y_0, z_0]$$

$$\underline{a * (x - x_0) + b * (y - y_0) + c * (z - z_0) = 0}$$

$$\therefore 4 * (x + 2) + 4 * (y - 0) - 3 * (z - 2) = 0$$

$$4x+4y-3z+14=0$$



Section 10.5: Problem 17

(1 pt)

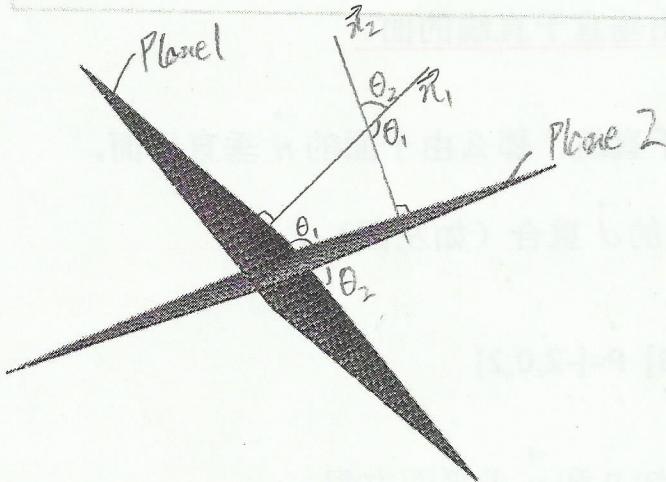
Find the angle of intersection of the plane $3x - 5y + 5z = 2$ with the plane $-5x + 4y + 5z = 0$.

Answer in radians:

1.4098

and in degrees:

80.7784

题型：根据 \vec{n} 求两面夹角（一般求锐角）

$$\vec{n}_1 = [3, -5, 5] \quad \vec{n}_2 = [-5, 4, 5]$$

$$\because \vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1| \cdot |\vec{n}_2| \cos \theta$$

$$\therefore \cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|}$$

$$= \frac{[3, -5, 5] \cdot [-5, 4, 5]}{|\vec{n}_1| \cdot |\vec{n}_2|}$$

$$= \frac{-15 + 20 + 25}{\sqrt{9+25+25} \cdot \sqrt{25+16+25}} = \frac{-10}{\sqrt{59} \cdot \sqrt{66}} = -0.16$$

$$= -0.16025147$$

$$\cos^{-1} \theta = 80.778^\circ = 80.778 \times \frac{\pi}{180} \text{ radians} = 1.409 \text{ rad}$$

10.5 16 ??? 2.

P18. Web

Find an equation of the plane consisting of all points that are equidistant from $(4, -3, -4)$ and $(-5, 5, 0)$. Page 7

$$\vec{n} = [-5, 5, 0] - [4, -3, -4] \\ = [-9, 8, 4]$$

$d_1 = d_2$

Suppose A is a point on the Plane.

$$A = \left[\frac{4+(-5)}{2}, \frac{-3+5}{2}, \frac{-4+0}{2} \right] = \left[-\frac{1}{2}, 1, -2 \right]$$

\therefore Plane is $\vec{n}(y - y_0) = 0$

$$-9(x + \frac{1}{2}) + 8(y - 1) + 4(z + 2) = 0$$

P19 Web 题型: 点到直线的距离

Find the distance from the point $(-3, -2, -5)$ to the plane $-3x - 4y + 2z = 7$

At first we need to find a point P on the plane.



$$\vec{PA} = [-2, -1, -5]$$

$$\vec{n} = [-3, -4, 2]$$

$P(-1, -1, 0)$ makes $-3x - 4y + 2z = 3 + 4 = 7 \Rightarrow P$ on the plane.

$$\because \text{Proj}_{\vec{n}} b = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a} : \vec{PA} = \text{Proj}_{\vec{n}} \vec{PA} = \frac{\vec{PA} \cdot \vec{n}}{|\vec{n}|^2} \vec{n} = \frac{[-3, -4, 2] \cdot [-2, -1, 5]}{9 + 16 + 4} [-3, -4, 2] \\ = \frac{6 + 4 - 10}{29} [-3, -4, 2] \\ = [0, 0, 0]$$

$$|\vec{PA}| = 0$$

Verification: plug P into $-3x - 4y + 2z = 7$: $-3(-1) - 4(-1) + 2(0) = 7$

P20

Page 7

Consider the planes $5x+11y+2z=1$ and $5x+2z=0$.

(a) Find the unique point P on the y-axis which is on both planes
 (on the y-axis means $x, z=0$).

$$\therefore 5x+11y+2z=1 \Rightarrow y=1$$

$$\therefore P[0, 1, 0]$$

(b) Find a unit vector u with positive first coordinate that is parallel to both planes.

$$\vec{n}_1 = [5, 1, 2] \quad \vec{n}_2 = [5, 0, 2]$$

$$\begin{vmatrix} i & j & k \\ 5 & 1 & 2 \\ 5 & 0 & 2 \end{vmatrix} = 1 \begin{vmatrix} i & k \\ 5 & 2 \end{vmatrix} - 1 \begin{vmatrix} j & k \\ 5 & 2 \end{vmatrix} + 1 \begin{vmatrix} i & j \\ 5 & 0 \end{vmatrix} = 2i - 5k$$

$$\therefore u = \frac{[2, 0, -5]}{\sqrt{4+25}} = \frac{[2, 0, -5]}{\sqrt{29}} = \left[\frac{2}{\sqrt{29}}, \frac{0}{\sqrt{29}}, \frac{-5}{\sqrt{29}} \right]$$

(c) Use parts (A)&(B) to find a vector equation for the line of intersection
 of the two planes, $r(t) =$

$$\begin{aligned} P + s \cdot u &= [0, 1, 0] + s \left[\frac{2}{\sqrt{29}}, 0, \frac{-5}{\sqrt{29}} \right] \\ &= \left[0 + s \frac{2}{\sqrt{29}}, 1 + 0s, 0 + \frac{-5}{\sqrt{29}}s \right] \\ &= \left(\frac{2}{\sqrt{29}}s \right)i + (1)j + \left(\frac{-5}{\sqrt{29}}s \right)k \end{aligned}$$



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Section 10.6: Problem 2

(1 pt)

Match the equation with its graph labeled A-H. You may click on any image to get a larger view.

Problems

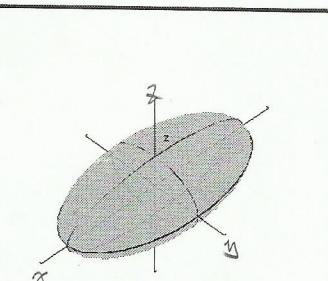
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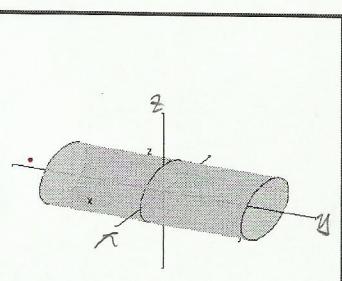
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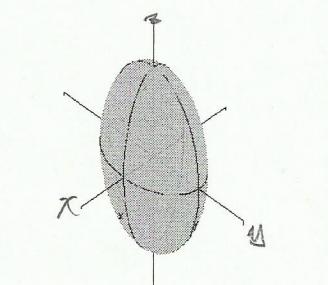
A.



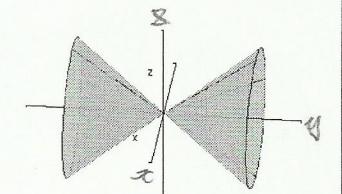
B.



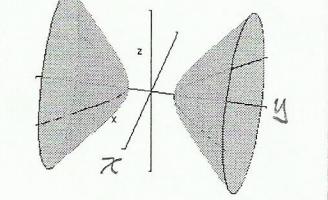
C.



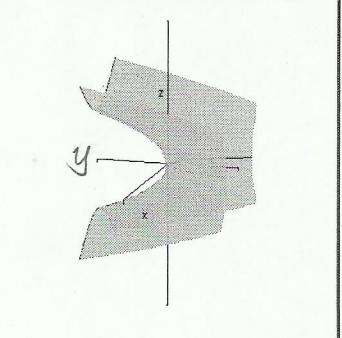
D.



E.



F.



B

$$1. \quad x^2 + 2z^2 = 1$$

A

$$2. \quad x^2 + 4y^2 + 9z^2 = 1$$

C

$$3. \quad 9x^2 + 4y^2 + z^2 = 1$$

D

$$4. \quad y^2 = x^2 + 2z^2$$

E

$$5. \quad -x^2 + y^2 - z^2 = 1$$

F

$$6. \quad y = x^2 - z^2$$

Note: You can earn partial credit on this problem.

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Fixed the intersecting curve of a paraboloid and a plane

Question details

Consider the paraboloid $x = 2z^2$. The plane $6x - 10y + z = 0$ cuts the paraboloid. Its intersection forms a curve.

Find the "natural" parametrization of this curve.

$\mathbf{Q(t)} = \langle x(t), y(t), z(t) \rangle$

$x(t) =$
 $y(t) =$
 $z(t) =$

Solve:

Comments (0)

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What are Karma Points?

Score Point Exchanges

Aug 16 5 events ago

Re: Find the intersecting curve of a paraboloid and a plane

Update: 20/2/13 Karma Points Expert

Another Detail:

Given equation of paraboloid is $z = x^2 + y^2$ and the plane is $6x - 10y + z = 0$

$$\Rightarrow z = -6x + 10y + 0$$

We have to find the intersection curve of these curves

$$x^2 + y^2 = -6x + 10y + 0$$

$$\Rightarrow x^2 + y^2 + 6x - 10y - 0 = 0$$

$$\Rightarrow x^2 + 6x + 9 + y^2 - 10y + 25 - 25 - 9 - 0 = 0$$

$$\Rightarrow (x+3)^2 + (y-5)^2 - 42 = 0$$

$$\Rightarrow (x+3)^2 + (y-5)^2 = 42$$

$$\Rightarrow (x+3)^2 + (y-5)^2 = 42$$

$$\Rightarrow \left(\frac{x+3}{\sqrt{42}}\right)^2 + \left(\frac{y-5}{\sqrt{42}}\right)^2 = 1$$

Comparing this equation with $\cos^2 t + \sin^2 t = 1$, we get

$$\frac{x+3}{\sqrt{42}} = \cos t, \frac{y-5}{\sqrt{42}} = \sin t$$

$$\Rightarrow x = -3 + \sqrt{42} \cos t, y = 5 + \sqrt{42} \sin t$$

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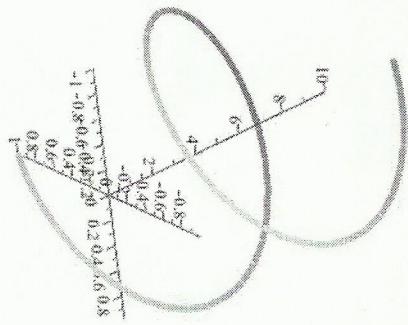
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The curve $\mathbf{C}(t) = (\cos t, \sin t, t)$ lies on which of the following surfaces.

Enter T or F depending on whether the statement is true or false.

(You must enter T or F -- True and False will not work.)

- | | |
|---|------------------------|
| F | 1. a plane |
| T | 2. a circular cylinder |
| F | 3. an ellipsoid |
| F | 4. a sphere |



Consider the paraboloid $z = x^2 + y^2$. The plane $2x - 5y + z - 10 = 0$ cuts the paraboloid, its intersection being a curve.

Find "the natural" parametrization of this curve.

Hint: The curve which is cut lies above a circle in the xy-plane which you should parametrize as a function of the variable t so that the circle is traversed counterclockwise exactly once as t goes from 0 to 2π , and the parameterization starts at the point on the circle with largest x coordinate. Using that as your starting point, give the parametrization of the curve on the surface.

$$\begin{aligned} \mathbf{c}(t) &= (x(t), y(t), z(t)), \text{ where} \\ x(t) &= \boxed{\quad} \\ y(t) &= \boxed{\quad} \\ z(t) &= \boxed{\quad} \end{aligned}$$

$$\begin{aligned} \therefore z &= x^2 + y^2 \\ \therefore x^2 + y^2 &= -2x + 5y + 10 \end{aligned}$$

$$\begin{aligned} x^2 + 2x + 1^2 + y^2 - 5y + 2.5^2 &= 3.75 \\ (x+1)^2 + (y-2.5)^2 &= 3.75 \\ \frac{(x+1)^2}{3.75} + \frac{(y-2.5)^2}{3.75} &= 1 \end{aligned}$$

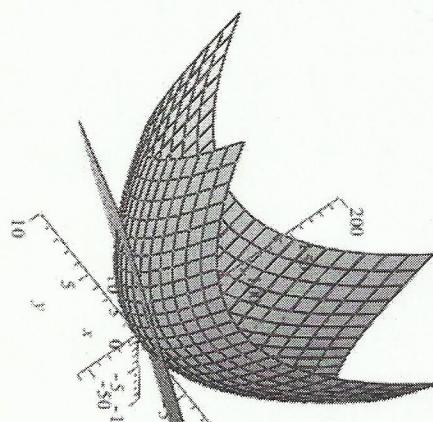
$$\therefore \cos t + \sin t = 1$$

$$\therefore \frac{x+1}{\sqrt{3.75}} = \cos \frac{y-2.5}{\sqrt{3.75}} = \sin t$$

$$\left\{ \begin{array}{l} x = \sqrt{3.75} \cos t - 1 \\ y = \sqrt{3.75} \sin t + 2.5 \end{array} \right.$$

$$\therefore z = -2\sqrt{3.75} \cos t + 10$$

$$\therefore z = -2\sqrt{3.75} \cos t + (-2\sqrt{3.75} \sin t + 2.5) + 10$$



Sec 10.8 P4

Consider the helix $\mathbf{r}(t) = (\cos(4t), \sin(4t), 3t)$. Compute at $t = \frac{\pi}{6}$. T.N.B

$$\text{tangent vector } \mathbf{T} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

principal unit normal vector $\mathbf{N}(t)$ (or simply unit normal)

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

Binormal vector:

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\begin{aligned}\mathbf{T} &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{[-\sin(4t) \cdot 4, \cos(4t) \cdot 4, 3]}{\sqrt{16\sin^2(4t) + 16\cos^2(4t) + 9}} = \frac{[-\sin(4t) \cdot 4, \cos(4t) \cdot 4, 3]}{\sqrt{16(\sin^2(4t) + \cos^2(4t)) + 9}} = \frac{[-\sin(4t) \cdot 4, \cos(4t) \cdot 4, 3]}{\sqrt{16+9}} \\ &= \frac{1}{5} [-4\sin(4t), 4\cos(4t), 3]\end{aligned}$$

$$\therefore \mathbf{T}\left(\frac{\pi}{6}\right) = \frac{1}{5} [-4(\sin\frac{2\pi}{3}), 4\cos(\frac{2\pi}{3}), 3] = \frac{1}{5} [-4 \times -\frac{\sqrt{3}}{2}, 4 \times -\frac{1}{2}, 3] = \frac{1}{5} [-2\sqrt{3}, -2, 3]$$

$$= \left[\frac{-2\sqrt{3}}{5}, -\frac{2}{5}, \frac{3}{5} \right]$$

$$\begin{aligned}\mathbf{N}(t) &= \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{\frac{1}{5} [-4\cos(4t) \cdot 4, -4\sin(4t) \cdot 4, 0]}{\sqrt{16^2[\cos^2(4t) + \sin^2(4t)] + 0^2}} = \frac{1}{16} [-16\cos(4t), -16\sin(4t), 0] \\ &= [-\cos(4t), -\sin(4t), 0]\end{aligned}$$

$$\therefore \mathbf{N}\left(\frac{\pi}{6}\right) = \left[-\cos\left(\frac{2\pi}{3}\right), -\sin\left(\frac{2\pi}{3}\right), 0 \right] = \left[\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0 \right]$$

$$\mathbf{B}(t) = \mathbf{N}(t) \times \mathbf{T}(t) = [0.519, 0.3, 0.8]$$

P1. Find the length of the given curve $\mathbf{r}(t) = (2t, 4\sin t, 4\cos t)$ where $-5 \leq t \leq 1$

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt = \int_a^b \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2} dt \quad \text{尺尺长}$$

$$L = \int_a^b \|\mathbf{r}'(t)\| dt$$

$$\mathbf{r}'(t) = [2, 4\cos(t), -4\sin(t)]$$

$$L = \int_{-5}^1 \sqrt{4 + 16\cos^2(t) + 16\sin^2(t)} dt = \int_{-5}^1 \sqrt{20} dt = \sqrt{20} \int_{-5}^1 dt = \sqrt{20} [t]_{-5}^1 = \sqrt{20}(1+5) = 6\sqrt{20}$$

2. Find the velocity and position vectors of a particle with acceleration $a(t) = [0, 0, 6]$, and initial conditions $v(0) = [-4, 4, -4]$ and $r(0) = [1, -4, 5]$.

$$v(t) = \int a(t) dt = \int [0, 0, 6] dt = [0, 0, 6t] + C$$

$$\therefore v(0) = [-4, 4, -4]$$

$$\therefore v(t) = \underline{[-4, 4, -4]} + \underline{[0, 0, 6t]} \xrightarrow{\text{key}} \underline{[-4, 4, -4 + 6t]} \quad \text{满足 } v(0) = [-4, 4, -4]$$

$$\therefore r(t) = \int v(t) dt = \int [-4, 4, -4 + 6t] dt = [-4t, 4t, -4t + 3t^2] + C$$

$$\therefore r(0) = [1, -4, 5]$$

$$r(t) \xrightarrow{\text{满足 }} r(0) = [1, -4, 5]$$

$$\therefore r(t) = [1, -4, 5] + [-4t, 4t, -4t + 3t^2] = [1 - 4t, -4 + 4t, 5 - 4t + 3t^2]$$

3. Given that the acceleration vector is $a(t) = (-4\cos(-2t))\mathbf{i} + (-4\sin(-2t))\mathbf{j} + (1t)\mathbf{k}$, the initial velocity is $v(0) = t\mathbf{k}$, and the initial position vector is $r(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$. $v(t), r(t)$

$$\begin{aligned} \int a(t) dt &= \int [-4\cos(-2t), -4\sin(-2t), t] dt = \left[\frac{-4\sin(-2t)}{2}, \frac{+4\cos(-2t)}{2}, \frac{t^2}{2} \right] \\ &= [2\sin(-2t), -2\cos(-2t), \frac{t^2}{2}] + C \end{aligned}$$

$$\therefore v(0) = [0, 0, 1]$$

$$\begin{cases} 2\sin(0) + C_1 = 0 \Rightarrow C_1 = 0 \\ -2\cos(0) + C_2 = 0 \Rightarrow C_2 = 2 \\ 0 + C_3 = 1 \Rightarrow C_3 = 1 \end{cases}$$

$$\therefore v(t) = [2\sin(-2t) + 0, -2\cos(-2t) + 2, \frac{t^2}{2} + 1]$$

$$\begin{aligned} \int v(t) dt &= \left[\frac{-2\cos(-2t)}{2}, \frac{-2\sin(-2t)}{2} + 2t, \frac{1}{2} \cdot \frac{t^3}{3} + t \right] \\ &= [\cos(-2t) + t, -\sin(-2t) + 2t, \frac{t^3}{6} + t] + C \end{aligned}$$

$$\therefore r(0) = [1, 1, 1]$$

$$\begin{cases} \cos(0) + 0 + C_1 = 1 \Rightarrow C_1 = 0 \\ -\sin(0) + 0 + C_2 = 1 \Rightarrow C_2 = 1 \\ 0 + 0 + C_3 = 1 \Rightarrow C_3 = 1 \end{cases}$$

$$\therefore r(t) = [\cos(-2t) + t, -\sin(-2t) + 2t + 1, \frac{t^3}{6} + t + 1]$$

6. A body of mass 9 kg moves in (counterclockwise) circular path of radius 2 meters, making one revolution every 11 seconds. You may assume the circle is in the xy-plane, and so you may ignore the third component.

A. Compute the centripetal force acting on the body.

$$\therefore \gamma(t) = a \cos(\omega t) \mathbf{i} + a \sin(\omega t) \mathbf{j} \quad a \text{ is radius.}$$

$$\therefore \gamma(t) = 2 \cos\left(\frac{2\pi}{11}t\right) \mathbf{i} + 2 \sin\left(\frac{2\pi}{11}t\right) \mathbf{j}$$

$$v(t) = \gamma'(t) = \left[-2 \sin\left(\frac{2\pi}{11}t\right) \times \frac{2\pi}{11} \mathbf{i} + \left[2 \cos\left(\frac{2\pi}{11}t\right) - \frac{2\pi}{11} \mathbf{j} \right] \right]$$

$$= \left[\frac{4\pi}{11} \sin\left(\frac{2\pi}{11}t\right), \frac{4\pi}{11} \cos\left(\frac{2\pi}{11}t\right) \right]$$

$$a(t) = v'(t) = \left[-\frac{4\pi}{11} \cos\left(\frac{2\pi}{11}t\right) \cdot \frac{2\pi}{11}, -\frac{4\pi}{11} \sin\left(\frac{2\pi}{11}t\right) \cdot \frac{2\pi}{11} \right]$$

$$= \left[2\left(\frac{2\pi}{11}\right)^2 \cos\left(\frac{2\pi}{11}t\right), 2\left(\frac{2\pi}{11}\right)^2 \sin\left(\frac{2\pi}{11}t\right) \right]$$

$$\therefore F = ma = 9 \cdot \left[2\left(\frac{2\pi}{11}\right)^2 \cos\left(\frac{2\pi}{11}t\right), 2\left(\frac{2\pi}{11}\right)^2 \sin\left(\frac{2\pi}{11}t\right) \right]$$

$$= \left[18\left(\frac{2\pi}{11}\right)^2 \cos\left(\frac{2\pi}{11}t\right), 18\left(\frac{2\pi}{11}\right)^2 \sin\left(\frac{2\pi}{11}t\right) \right]$$

B. Compute the magnitude of that force.

$$F = \frac{mv^2}{r} = \frac{9 \times \frac{(2\pi \times 2)^2}{2}}{2} = \frac{9 \times (4\pi)^2}{2} \quad N$$

4. The position function of a particle is given by $r(t) = [3t^2, -4t, t^2+2t]$
At what time is the speed minimum?

$$r'(t) = [6t, -4, 2t+2] \quad \text{IV}$$

$$\|r'(t)\| = \sqrt{36t^2 + 16 + 4t^2 + 8t + 4} = \sqrt{40t^2 + 8t + 20} = 2\sqrt{10t^2 + 2t + 5}$$

$\|r'(t)\|$ be min, $10t^2 + 2t + 5$ should be min key

$$\therefore (10t^2 + 2t + 5)' = 20t + 2$$

$$\text{Let } 20t + 2 = 0 \Rightarrow t = -\frac{1}{10} \rightarrow \min \quad \text{key}$$

\therefore at $t = -\frac{1}{10}$ speed min.

5. A dense particle with mass 10kg follows the path $r(t) = [\sin(8t), \cos(9t), 2t^{\frac{3}{2}}]$ with units in meters and seconds. What force acts on the mass at $t=0$?

$$\therefore F = m \alpha$$

$$V(t) = r'(t) = [\cos(8t) \cdot 8, -\sin(9t) \cdot 9, 2 \times \frac{3}{2} t^{\frac{1}{2}}] \\ = [8\cos(8t), -9\sin(9t), 7 + \frac{5}{2}t^{\frac{1}{2}}]$$

$$\alpha(t) = V'(t) = [-8\sin(8t) \cdot 8, -9\cos(9t) \cdot 9, 7 \times \frac{5}{2}t^{\frac{3}{2}}] \\ = [-64\sin(8t), -81\cos(9t), \frac{35}{2}t^{\frac{3}{2}}]$$

$$\therefore F = m \alpha = 10 \cdot \alpha(0) = 10 \cdot [0, -81, 0] = [0, -810, 0] \text{ kg m/s}^2 \quad \text{key}$$



Web

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Sec 11.3 P5

Find the first partial derivatives of $f(x, y, z) = z \arctan(\frac{y}{x})$ at the point $(5, 5, 1)$.

A. $\frac{\partial f}{\partial x}(5, 5, 1) =$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \quad \text{key}$$

$$\frac{\partial f}{\partial x} = z \frac{1}{1+(\frac{y}{x})^2} y x (-1) x^{-2} = -\frac{zy}{1+(\frac{y}{x})^2} x \frac{1}{x^2} = \frac{-zy}{[1+(\frac{y}{x})^2]x^2}$$

$$\therefore x=5 \quad y=5 \quad z=1$$

$$\therefore \frac{\partial f}{\partial x}(5, 5, 1) = \frac{-1 \times 5}{[1+(\frac{5}{5})^2] \times 5^2} = \frac{-5}{2 \times 25} = -\frac{5}{50} = -\frac{1}{10}$$

B. $\frac{\partial f}{\partial y}(5, 5, 1) =$

$$\frac{\partial f}{\partial y} = z \frac{1}{1+(\frac{y}{x})^2} \frac{1}{x}$$

$$\frac{\partial f}{\partial y}(5, 5, 1) = 1 \cdot \frac{1}{1+(\frac{5}{5})^2} \frac{1}{5} = 1 \times \frac{1}{2} \times \frac{1}{5} = \frac{1}{10}$$

C. $\frac{\partial f}{\partial z}(5, 5, 1) =$

$$\frac{\partial f}{\partial z} = \arctan(\frac{y}{x})$$

$$\frac{\partial f}{\partial z}(5, 5, 1) = \arctan(\frac{5}{5}) = \arctan(1)$$

P6. The gas law for a fixed mass m of an ideal gas at absolute temperature T , pressure P , and volume V is $PV = mRT$, where R is the gas constant. Find the partial derivatives.

$$\frac{\partial P}{\partial V} = ? \quad \because PV = mRT \Rightarrow \frac{\partial P}{\partial V} = mRT \times (-1)V^{-2} = -\frac{mRT}{V^2}$$

$$\frac{\partial V}{\partial T} = ? \quad \because PV = mRT \Rightarrow \frac{\partial V}{\partial T} = \frac{mR}{P}$$

$$\frac{\partial T}{\partial P} = ? \quad \because PV = mRT \Rightarrow \frac{\partial T}{\partial P} = \frac{V}{mR}$$

$$\frac{\partial P}{\partial V} \frac{\partial V}{\partial T} \frac{\partial T}{\partial P} = ? \quad -\frac{mRT}{V^2} \times \frac{mR}{P} \times \frac{V}{mR} = -\frac{TmR}{VP} \quad \therefore P = \frac{mRT}{V} \quad \therefore = -\frac{TmR}{V} \times \frac{V}{mR} = -1$$

key

不能直接化简 ~~$\frac{\partial P}{\partial V} \frac{\partial V}{\partial T} \frac{\partial T}{\partial P}$~~ = 1 立是错的

P7. Find the partial derivatives of the function

$$f(x, y) = xy e^{2y}$$

You should as a by product verify that the function f satisfies Clairaut's theorem.

$$f_x(x, y) = ye^{2y}$$

$$f_y(x, y) = x(e^{2y} + ye^{2y} \cdot 2) = x(e^{2y} + 2ye^{2y})$$

$$f_{xy}(x, y) = ?$$

$$f_{yx}(x, y) = ?$$

If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

$$\{ f_{xy}(x, y) = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (ye^{2y}) = e^{2y} + ye^{2y} \cdot 2 = e^{2y} + 2ye^{2y}$$

$$f_{yx}(x, y) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} [x(e^{2y} + ye^{2y} \cdot 2)] = e^{2y} + 2ye^{2y}$$

Clairaut's Theorem:

Suppose f is defined on a disk D that contains the point (a, b) .

If the function f_{xy} and f_{yx} are both continuous on D , then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

$$f_{xyy} = f_{yxy} = f_{yyx}$$

P9. If $\sin(-5x+z)=0$, find the first partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(0, 0, 0)$.

$$A. \frac{\partial z}{\partial x}(0, 0, 0) = ? \quad B. \frac{\partial z}{\partial y}(0, 0, 0) = ?$$

$$A. \sin(-5x+z)=0$$

两边同时求导

$\frac{\partial z}{\partial x}$ 表示 z 是 x 的
函数。

$$-\frac{\partial z}{\partial x} = -5 + \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = 5$$

$$B. \cos(-5x+z) \times (0 + \frac{\partial z}{\partial y}) = 0$$

$$\frac{\partial z}{\partial y} = 0$$

Web 11.4 Ps

Find the linearization of the function $z = \sqrt{xy}$ at the point $(3, 4)$.

$$L(x, y) = ?$$

$$f(x, y) \approx L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \text{ key tangent plane Pg 18}$$

$$f(a, b) = \sqrt{ab} = \sqrt{3 \cdot 4} = 2\sqrt{3} = 21$$

$$f_x(a, b) = \frac{\partial}{\partial x} \sqrt{xy} \Big|_{(3, 4)} = \frac{1}{2\sqrt{xy}} \cdot y = \frac{1}{2\sqrt{3 \cdot 4}} \cdot 4 = \frac{1}{2\sqrt{3}} = \frac{1}{2}\sqrt{3}$$

$$f_y(a, b) = \frac{\partial}{\partial y} \sqrt{xy} \Big|_{(3, 4)} = \frac{1}{2\sqrt{xy}} \cdot x = \frac{1}{2\sqrt{3 \cdot 4}} \cdot 3 = \frac{3}{2\sqrt{3}} = \frac{3}{4}\sqrt{3}$$

$$\therefore L(x, y) = 21 + \frac{1}{2}\sqrt{3}(x - 3) + \frac{3}{4}\sqrt{3}(y - 4)$$

4. Find the linearization of the function $f(x, y) = \sqrt{17 - x^2 - y^2}$ at the point $(2, -2)$

$$L(x, y) = ?$$

$$f(x, y) = \sqrt{17 - x^2 - y^2} = \sqrt{9} = 3$$

$$f_x(x, y) = \frac{1}{2} \times \frac{1}{\sqrt{17-x^2-y^2}} \times (0 - 2x - 0) = \frac{-x}{\sqrt{17-x^2-y^2}} = \frac{-2}{3}$$

$$f_y(x, y) = \frac{1}{2} \times \frac{1}{\sqrt{17-x^2-y^2}} \times (0 - 0 - 2y) = \frac{-y}{\sqrt{17-x^2-y^2}} = \frac{+2}{3}$$

$$\therefore L(x, y) = 3 + \frac{2}{3}(x - 2) + (-\frac{1}{3}(y - (-2))) = 3 - \frac{2}{3}(x - 2) + \frac{2}{3}(y + 2)$$

Use the linear approximation to estimate the value of $f(1.9, -1.9) = ?$

$$\therefore L(x, y) = 3 - \frac{2}{3}(x - 2) + \frac{2}{3}(y + 2)$$

$$\therefore L(1.9, -1.9) = 3 - \frac{2}{3}(1.9 - 2) + \frac{2}{3}(-1.9 + 2) = 3 - \frac{2}{3} \times (-0.1) + \frac{2}{3} \times 0.1 = 3 + \frac{2}{15} \\ = 3.13333$$

6. Find the differential of the function $w = x^5 \sin(y^4 z^7)$

$$dw = ?$$

$$dw = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \text{ key Pg 18 应用题核心公式}$$

$$f_x(x, y) = 5x^4 \sin(y^4 z^7)$$

$$f_y(x, y) = x^5 (\cos(y^4 z^7)) z^7 4x y^3 = 4x^5 \cos(y^4 z^7) z^7 y^3$$

$$f_z(x, y) = x^5 (\cos(y^4 z^7)) y^4 x z^6 = 7x^5 \cos(y^4 z^7) y^4 z^6$$

$$\therefore dw = 5x^4 \sin(y^4 z^7) dx + 4x^5 \cos(y^4 z^7) z^7 y^3 dy + 7x^5 \cos(y^4 z^7) y^4 z^6 dz$$

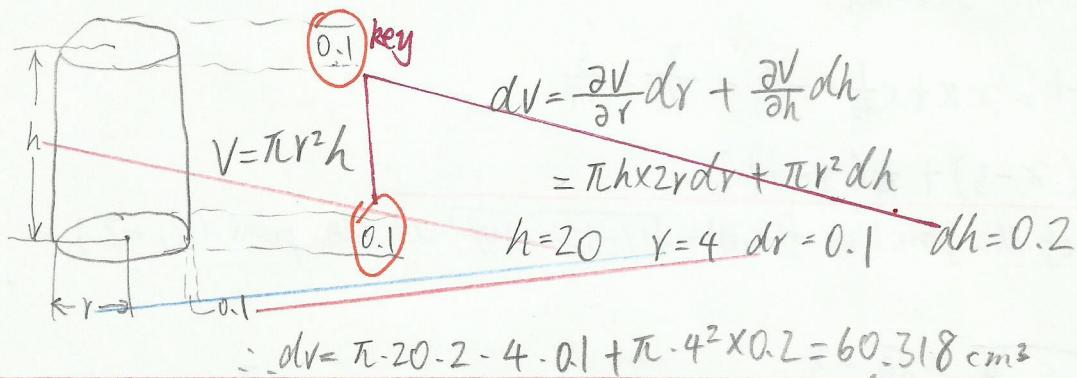
Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

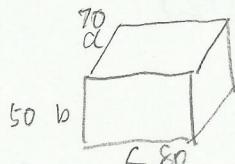
7. Use differentials to estimate the amount of material in a closed cylindrical can that is 20cm high and 8cm in diameter if the metal in the top and bottom is 0.1cm thick, and the metal in the sides is 0.1 cm thick. Note, you are approximating the volume of metal which makes up the can, not the volume it encloses.

The differential for the volume is

key



8. The dimensions of a closed rectangular box are measured as 70 cm, 80 cm and 50cm, respectively, with the error in each measurement at most 0.2cm. Use differentials to estimate the maximum error in calculating the surface area of the box. 題型: 求Error Calculus 1 P10



$$S = 2ab + 2bc + 2ca$$

$$\begin{aligned} \text{Error} &= dS = (2b+2c)da + (2a+2c)db + (2b+2a)dc \\ &= (2 \times 50 + 2 \times 80) \times 0.2 + (2 \times 70 + 2 \times 80) \times 0.2 + (2 \times 50 + 2 \times 70) \times 0.2 \\ &= 52 + 60 + 48 = 160 \text{ cm}^2 \end{aligned}$$

Web.

1. Suppose $w = \frac{x}{y} + \frac{y}{z}$, where $x = e^t$, $y = 2 + \sin(2t)$, and $z = 2 + \cos(3t)$

A. Use the chain rule to find $\frac{dw}{dt}$ as function of x, y, z and t . Do not rewrite x, y and z in terms of t , and do not rewrite e^t as x .

key The chain rule

Suppose that u is a differentiable function of the n variables x_1, x_2, \dots, x_n , and each x_i is a differentiable function of the m variables t_1, t_2, \dots, t_m . Then u is a function of t_1, t_2, \dots, t_m and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i} \quad \text{for each } i=1, 2, \dots, m.$$

key

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

$$\therefore \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

$$\frac{\partial w}{\partial x} = \frac{1}{y} \quad \frac{dx}{dt} = e^t \quad \frac{\partial w}{\partial y} = x - 1 \times y^{-2} = \frac{-x}{y^2} + \frac{1}{2} \quad \frac{dy}{dt} = \cos(2t) \cdot 2 = 2 \cos(2t)$$

$$\frac{\partial w}{\partial z} = y - 1 \times z^{-2} = \frac{-y}{z^2} \quad \frac{dz}{dt} = -\sin(3t) \cdot 3 = -3 \sin(3t)$$

$$\therefore \frac{dw}{dt} = \frac{1}{y} e^t + \frac{1}{2} \times \frac{-x}{y^2} \times 2 \cos(2t) + \frac{-y}{z^2} \times (-3 \sin(3t))$$

B. Use part A to evaluate $\frac{dw}{dt}$ when $t=0$.

$$\begin{aligned} \frac{dw}{dt} &= \frac{1}{2 + \sin(2t)} e^t + \left(\frac{-et}{(2 + \sin(2t))^2} + \frac{1}{2 + \cos(3t)} \right) 2 \cos(2t)H - \frac{(2 + \sin(2t)) \times (-3 \sin(3t))}{(2 + \cos(3t))^2} \\ &= \frac{1}{2} + \left(-\frac{1}{2} + \frac{1}{3} \right) 2 + \frac{2}{3^2} \times 0 \\ &= 0.6666 \end{aligned}$$

2. Suppose $z = x^2 \sin y$, $x = -5s^2 + 3t^2$, $y = 8st$.

A. Use the chain rule to find $\frac{\partial z}{\partial s}$, and $\frac{\partial z}{\partial t}$ as function of x, y, s and t .

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \quad \text{key}$$

$$\frac{\partial z}{\partial x} = \sin y \times 2x \quad \frac{\partial x}{\partial s} = -10s \quad \frac{\partial z}{\partial y} = x^2 \cos(y) \quad \frac{\partial y}{\partial s} = -8t$$

$$\therefore \frac{\partial z}{\partial s} = 2x \sin y (-10s) + x^2 \cos(y) (-8t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

$$\frac{\partial x}{\partial t} = 6t \quad \frac{\partial y}{\partial t} = -8s$$

$$\therefore \frac{\partial z}{\partial t} = \sin y \times 2x \cdot 6t + x^2 \cos(y) \cdot (-8s)$$

4 Let $w = \frac{\partial}{\partial s} (3y - xz - xz^2)$, $x=st$, $y=e^{st}$, $z=t^2$

Compute

$$\frac{\partial w}{\partial s}(2, -4)$$

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s} \\ &= (3y - z)xt + (3x - z)e^{st}t + (0 - y - z)0 \\ &= 3(3y - z) + e^{st}t(3x - z)\end{aligned}$$

$$\begin{aligned}-\therefore \frac{\partial w}{\partial s}(2, -4) &= (3xe^{st} - t^2) + e^{st}t(3xst - t^2) \\ &= (3xe^{-8} - 16) + e^{-8}t(-4)(3x(-8) - 16) \\ &= (3e^{-8} - 16) + (3x(-8) - 16)e^{-8}(-4)\end{aligned}$$

5. Let $W(s, t) = F(u(s, t), v(s, t))$ where

$$u(1, 0) = 8, u_s(1, 0) = 8, u_t(1, 0) = -8$$

$$v(1, 0) = 5, v_s(1, 0) = 4, v_t(1, 0) = -3$$

$$F_u(8, 5) = -4, F_v(8, 5) = 8$$

$$W_s(1, 0) = \frac{\partial W}{\partial s} = \frac{\partial W}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial W}{\partial v} \frac{\partial v}{\partial s} = F_u \times u_s + F_v \times v_s = -4 \times 8 + 8 \times 4 = 0$$

$$W_t(1, 0) = \frac{\partial W}{\partial t} = \frac{\partial W}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial W}{\partial v} \frac{\partial v}{\partial t} = F_u \times u_t + F_v \times v_t = -4 \times -8 + 8 \times -3 = -8$$

6. Consider the curve $x^2 + 5xy + y^6 = 7$

The equation of the tangent line to the curve at the point $(1, 1)$ has the form $y = mx + b$ where $m = ?$ and $b = ?$

$$2x + 5y + 5xy' + 6y^5y' = 0 \quad \text{两边同时求导}$$

$$2x + 5y + y'(5x + 6y^5) = 0$$

$$y' = \frac{-2x - 5y}{5x + 6y^5} = \frac{-2 - 5}{5 + 6} = -\frac{7}{11}$$

$$\therefore y = mx + b \quad \text{any } y = 1 \quad x = 1$$

$$\therefore 1 = -\frac{7}{11}x + b$$

$$b = 1 + \frac{7}{11}$$

7. Consider the surface $F(x, y, z) = x^7 z^7 + \sin(y^8 z^7) + 5 = 0$
 Find the following partial derivatives.

$$\frac{\partial z}{\partial x} = ? \quad \frac{\partial z}{\partial y} = ? \quad \text{给定等式, 两边同时求导.}$$

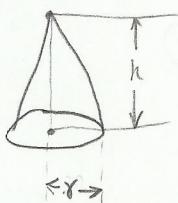
key For $\frac{\partial z}{\partial x}$, z is含x的表达式.

$$7x^6 z^7 + x^7 7z^6 \frac{\partial z}{\partial x} + \cos(y^8 z^7) \cdot y^8 7z^6 \frac{\partial z}{\partial x} = 0$$

$$7x^6 z^7 + \frac{\partial z}{\partial x} (x^7 7z^6 + \cos(y^8 z^7) \cdot y^8 \cdot 7z^6) = 0$$

$$\frac{\partial z}{\partial x} = \frac{-7x^6 z^7}{x^7 7z^6 + \cos(y^8 z^7) y^8 \cdot 7z^6}$$

key P8 The radius of a right circular cone is increasing at a rate of 5 inches per second and is decreasing at a rate of 3 inches per second. At what rate is the volume of the cone changing when the radius is 20 inches and height is 20 inches?



$$V = \frac{1}{3}\pi r^2 \cdot h$$

$$dV = \left| \frac{\partial V}{\partial r} \right| \frac{dr}{dt} + \left| \frac{\partial V}{\partial h} \right| \frac{dh}{dt} \quad dt = 1$$

$$\therefore dV = \left| \frac{\partial V}{\partial r} \right| dr + \left| \frac{\partial V}{\partial h} \right| dh$$

$$= \left(\frac{1}{3}\pi h \times 2r \right) dr + \left(\frac{1}{3}\pi r^2 \right) dh$$

$$h = 20 \quad r = 20 \quad dr = 5 \quad dh = -3$$

$$\therefore dV = \frac{1}{3}\pi \times 20 \times 2 \times 20 \times 5 + \frac{1}{3} \times \pi \times 20^2 \times (-3) = \left(\frac{2800}{3} \right) \pi$$

key P9. In a simple electric circuit, Ohm's law states that $V = IR$, where V is the voltage, I is the current in amperes, and R is the resistance in ohms. Assume that, as the battery wears out, the voltage decreases at 0.04 v per second and, as the resistor heats up, the resistance is increasing at 0.03 ohms per second. When the resistance is 100 ohms and the current is 0.01 amperes, at what rate is the current changing?

$$I = \frac{V}{R} \Rightarrow dI = \left| \frac{\partial I}{\partial V} \right| \cdot \frac{dV}{dt} + \left| \frac{\partial I}{\partial R} \right| \cdot \frac{dR}{dt} \quad dt = 1$$

$$\therefore dI = \left| \frac{\partial I}{\partial V} \right| \cdot dV + \left| \frac{\partial I}{\partial R} \right| dR$$

$$= \frac{1}{R} dV + V \times -1 \times R^{-2} dR$$

$$\therefore R = 100 \quad I = 0.01 \quad dV = 0.04 \quad dR = -0.03 \quad V = IR = 100 \times 0.01 = 1$$

$$\therefore dI = \frac{1}{100} \times 0.04 - 1 \times \frac{1}{100^2} \times 0.03 = -0.000403 \text{ A/s}$$

Web P2. Find the directional derivative of $f(x, y, z) = x^3 - x^2 y$ at the point $(-3, 1, 3)$ in the direction of the vector $V = \langle -4, -1, 3 \rangle$

Theorem If f is a differentiable function of x and y , then f has a directional derivative in the direction of any unit vector $u = \langle a, b \rangle$ and

$$\nabla f(x, y) = f_x(x, y) \mathbf{i} + f_y(x, y) \mathbf{j} = \nabla w \cdot \frac{V}{\|V\|}$$

$$\nabla f = [f_x, f_y, f_z] = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

Solution:

$$\frac{\partial w}{\partial x} = -2xy \quad \frac{\partial w}{\partial y} = -x^2 \quad \frac{\partial w}{\partial z} = 3z^2$$

$$\therefore \nabla w = \left[\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z} \right] = [-2xy, -x^2, 3z^2]$$

and (at point $(-3, 1, 3)$)

$$\therefore \nabla w = [6, -9, 27]$$

$$\frac{V}{\|V\|} = \frac{[-4, -1, 3]}{\sqrt{(-4)^2 + (-1)^2 + 3^2}} = \frac{[-4, -1, 3]}{\sqrt{16 + 1 + 9}} = \frac{[-4, -1, 3]}{\sqrt{26}}$$

$$\therefore \text{directional derivative: } \nabla w \cdot \frac{V}{\|V\|} = [6, -9, 27] \cdot \frac{[-4, -1, 3]}{\sqrt{26}} = 12.9437$$

* P3 Find the directional derivative of $f(x, y) = x^2 y^3 + 2x^4 y$ at the point $(1, -5)$ in the direction $\theta = 5\pi/6$. The gradient of f is:

$\nabla f = ? \quad \nabla f(1, -5) = ? \quad \text{The directional derivative is: ?}$

$$\nabla f(x, y) = \nabla f(x, y, z) \cdot u \quad u = \frac{\vec{v}}{\|\vec{v}\|} \quad \vec{v} = [\cos \theta, \sin \theta] \quad \text{key!}$$

$$\frac{\partial w}{\partial x} = 2y^3x + 8x^3y \quad \frac{\partial w}{\partial y} = 3x^2y^2 + 2x^4$$

$$\therefore \nabla f = [2xy^3 + 8x^3y, 3x^2y^2 + 2x^4]$$

and at point $(1, -5)$

$$\therefore \nabla f(1, -5) = [-290, 77]$$

For directional derivative:

$$u = \frac{V}{\|V\|} = \frac{[\cos \frac{5\pi}{6}, \sin \frac{5\pi}{6}]}{\sqrt{(\cos \frac{5\pi}{6})^2 + (\sin \frac{5\pi}{6})^2}} = \frac{[-0.866025, 0.5]}{\sqrt{0.749999 + 0.25}} = \frac{[-0.866025, 0.5]}{0.99999} \approx [-0.866025, 0.5]$$

$$\therefore \text{directional derivative: } \nabla f(1, -5) \cdot u = [-290, 77] \cdot [-0.866025, 0.5] = 289.6474$$

- key!!!** 4. Find the maximum rate of change of $f(x, y) = \ln(x^2 + y^2)$ at the point $(-1, -1)$ and the direction in which it occurs.

Maximum rate of change = ?

题型

* maximum rate of change
direction in which it occurs.

Direction (unit vector) in which it occurs = ?

Theorem

Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_u f(x)$ is $|\nabla f(x)|$ and it occurs when u has the same direction as the gradient vector $\nabla f(x)$.

Book P 639 key!!!

$$\frac{\partial w}{\partial x} = \frac{1}{x^2 + y^2} \times 2x = \frac{2x}{x^2 + y^2} \quad \frac{\partial w}{\partial y} = \frac{2y}{x^2 + y^2}$$

$$\therefore \nabla f = \left[\frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right]$$

and : at $(-1, -1)$

$$\therefore \nabla f = [-1, -1]$$

∴ Maximum rate of change is $|\nabla f| = \sqrt{1^2 + 1^2} = \sqrt{2}$

Direction (unit vector) in which it occurs:

$$u = \frac{[-1, -1]}{\sqrt{2}} = \left[-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right]$$

7. Suppose that you are climbing a hill whose shape is given by $z = 854 - 0.05x^2 - 0.06y^2$, and that you are at the point $(100, 30, 300)$.

- (1) In which direction (unit vector) should you proceed initially in order to reach the top of the hill fastest?

- (2) If you climb in that direction, at what angle above the horizontal will you be climbing initially (radian measure)?

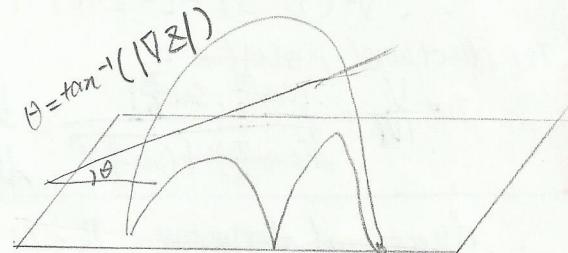
(3) For $x-y$ plane:

$$\frac{\partial z}{\partial x} = -0.1x \quad \frac{\partial z}{\partial y} = -0.12y$$

$$\therefore \nabla z = [-0.1x, -0.12y]$$

∴ at $[100, 30]$

$$\therefore \nabla z = \frac{[-10, -3.6]}{\sqrt{10^2 + 3.6^2}}$$



$$(2) \theta = \tan^{-1}(|\nabla z|) = \tan^{-1}(\sqrt{10^2 + 3.6^2}) = \tan^{-1}(10.6282642) = 84.624936^\circ$$

$$= 84.624936 \times \frac{\pi}{180} \text{ rad}$$

key! 非常重要

? Why?

Pg Find equations of the tangent plane and normal line to the surface $x = 5y^2 + 4z^2 - 276$ at the point $(4, 6, -5)$.

Tangent Plane:

$$\frac{\partial x}{\partial y} = 10y \quad \frac{\partial x}{\partial z} = 8z$$

\therefore at point $(4, 6, -5)$

$$\therefore \frac{\partial x}{\partial y} = 10 \times 6 = 60 \quad \frac{\partial x}{\partial z} = 8z = 8 \times -5 = -40$$

Tangent Plane:

$$(x - 4) = 60(y - 6) + (-40)(z - (-5))$$

Simplify: $x - 60y + 40z + 556 = 0$

Normal line: Normal vector

在平面上练习 $\langle 4, 6, -5 \rangle + t[1, -60, 40]$

题型: 求 Tangent Plane

Normal Line

Pg. Find equation of the tangent plane and normal line to the surface $z = 9e^y \cos z - 9$ at the point $(9, 0, 0)$. Tangent Plane: 题型: 非常重要, 隐函数求 tangent plane.

$$z = 9e^y \cos z - 9$$

$$\frac{\partial z}{\partial x} = e^y \cos z + 9e^y (-\sin z) \frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial y} = 9 [e^y \cos z + e^y (-\sin z) \frac{\partial z}{\partial y}]$$

\therefore at the point $(9, 0, 0)$

$$\frac{\partial z}{\partial x} = 1 + 0 = 1$$

$$\frac{\partial z}{\partial y} = 9(1+0) = 9$$

$\therefore (z - 0) = 1(x - 9) + 9(y - 0)$

$$z = x - 9 + 9y$$

$$z - x - 9y + 9 = 0$$

★ Second Derivatives Test

Suppose the second partial derivatives of f are continuous on a disk with center (a, b) , and suppose that $f_{xx}(a, b) = 0$ and $f_{yy}(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

- (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- (c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

Note 1 In case (c) the point (a, b) is called a saddle point of f and the graph of f crosses its tangent plane at (a, b) .

Note 2 If $D=0$, the test gives no information: f could have a local maximum or local minimum at (a, b) or (a, b) could be a saddle point of f .

Note 3 To remember the formula for D it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - [f_{xy}]^2$$

P4. Suppose $f(x, y) = x^2 + y^2 - 10x - 4y + 5$

(A) How many critical points does f have in \mathbb{R}^2 ? Find critical point

$$w = f(x, y) = x^2 + y^2 - 10x - 4y + 5$$

$$\frac{\partial w}{\partial x} = 2x - 10 \quad \text{Let } \frac{\partial w}{\partial x} = 0 \Rightarrow x = 5$$

$$\frac{\partial w}{\partial y} = 2y - 4 \quad \text{Let } \frac{\partial w}{\partial y} = 0 \Rightarrow y = 2$$

∴ Critical point is $[5, 2]$ \Rightarrow has only one critical point.

(B) If there is a local minimum, what is the value of the discriminant D at that point?

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - [f_{xy}]^2 \quad \text{determine local min or max}$$

$$\therefore f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = 0$$

$$\therefore D = 4 > 0 \quad \text{and } f_{xx} > 0$$

∴ $[5, 2]$ is a local min.

(F) What is the min. value of f on \mathbb{R}^2 ? Find local min

$$f(5, 2) = 5^2 + 2^2 - 10 \cdot 5 - 4 \cdot 2 + 5 = 25 + 4 - 50 - 8 + 5 = -24$$

P5. Consider the function

$$f(x, y) = \sqrt{x} - y^2 - 4x + 15y$$

Find and classify all critical points of the function. If there are more blanks than critical points, leave the remaining entries blank.

$$f_x = \frac{1}{2}y\sqrt{x}^{-\frac{1}{2}} - 4$$

$$f_y = \sqrt{x} - 2y + 15$$

$$f_{xx} = \frac{1}{2}y\sqrt{x}(-\frac{1}{2})x^{-\frac{3}{2}}$$

$$f_{xy} = \frac{1}{2}\sqrt{x}^{-\frac{1}{2}}$$

$$f_{yy} = -2$$

For Critical points: # Critical key method.

$$\left\{ \begin{array}{l} f_x = \frac{1}{2}y\sqrt{x}^{-\frac{1}{2}} - 4 = 0 \\ f_y = \sqrt{x} - 2y + 15 = 0 \end{array} \right. \quad (1)$$

$$\left\{ \begin{array}{l} f_x = \frac{1}{2}y\sqrt{x}^{-\frac{1}{2}} - 4 = 0 \\ f_y = \sqrt{x} - 2y + 15 = 0 \end{array} \right. \quad (2)$$

from 2:

$$2y = \sqrt{x} + 15$$

$$y = \frac{\sqrt{x} + 15}{2} \quad (3)$$

Plug (3) into (1):

$$\frac{1}{2}\left(\frac{\sqrt{x} + 15}{2}\right)\sqrt{x} - 4 = 0$$

$$\frac{\sqrt{x} + 15}{4\sqrt{x}} - 4 = 0$$

$$\frac{\sqrt{x}}{4\sqrt{x}} + \frac{15}{4\sqrt{x}} - 4 = 0$$

$$\frac{1}{4} + \frac{15}{4\sqrt{x}} = 4$$

$$1 + \frac{15}{\sqrt{x}} = 16$$

$$\frac{15}{\sqrt{x}} = 15$$

$$\sqrt{x} = 1$$

$$\therefore x = 1 \quad (4)$$

Plug (4) into (3):

$$y = \frac{\sqrt{1} + 15}{2} = \frac{16}{2} = 8$$

\therefore critical point is

$$[1, 8]$$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

$$f_{xx} = \frac{1}{2}y\sqrt{x}(-\frac{1}{2})x^{-\frac{3}{2}} = \frac{1}{2} \times 8 \times (-\frac{1}{2}) \times 1^{-\frac{3}{2}} = -2$$

$$f_{yy} = -2$$

$$f_{xy} = \frac{1}{2}\sqrt{x}^{-\frac{1}{2}} = \frac{1}{2}1^{-\frac{1}{2}} = \frac{1}{2}$$

$\therefore D = 4 - \frac{1}{4} > 0$ and $f_{xx} < 0$

$\therefore [1, 8]$ is local maximum.

7. Find the absolute maximum and absolute minimum of the function $f(x, y) = xy - y - x + 1$ on the region on or above $y = x^2$ and on or below $y = 6$. 题型：区域极值

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* To find the absolute maximum and minimum value of a continuous function f on a closed, bounded set D :

1. Find the values of f at the critical points of f in D .

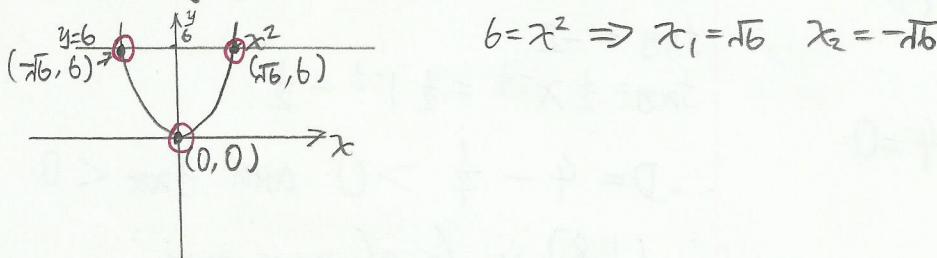
2. Find the extreme value of f on the boundary of D .

3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

$$\begin{aligned}\frac{\partial f}{\partial x} &= y - 1 \\ \frac{\partial f}{\partial y} &= x - 1\end{aligned}\Rightarrow \text{critical point } [1, 1]$$

$$D = \left| \begin{matrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{matrix} \right| = f_{xx}f_{yy} - (f_{xy})^2 = 0 - 1^2 = -1 < 0 \Rightarrow (1, 1) \text{ is not min or max}$$

From $y = x^2$ $y = 6$



$$6 = x^2 \Rightarrow x_1 = \sqrt{6}, x_2 = -\sqrt{6}$$

$$\text{When } [-\sqrt{6}, 6] \\ f(-\sqrt{6}, 6) = -17.24$$

$$\text{When } [\sqrt{6}, 6] \\ f(\sqrt{6}, 6) = 7.24$$

$$\text{When } [0, 0] \\ f(0, 0) = 1$$

- Absolute minimum value is -17.24 attained at $[-\sqrt{6}, 6]$
- Absolute maximum value is 7.24 attained at $[\sqrt{6}, 6]$

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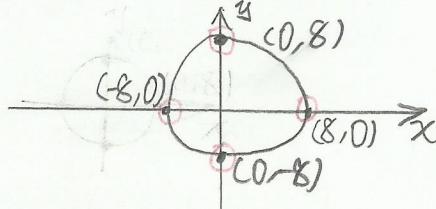
8. Find the absolute maximum and absolute minimum of the function $f(x, y) = 2x^3 + y^4$ on the region $\{(x, y) | x^2 + y^2 \leq 64\}$.

题型 区域极限

同类题 11.7. 7

$$\begin{aligned}\frac{\partial w}{\partial x} &= 6x^2 \\ \frac{\partial w}{\partial y} &= 4y^3\end{aligned} \Rightarrow \text{critical point } [0, 0]$$

For $x^2 + y^2 \leq 64 = 8^2$



$$f(8, 0) = 2 \times 8^3 = 1024$$

$$f(-8, 0) = 2 \times (-8)^3 = -1024$$

$$f(0, 8) = 8^4 = 4096$$

$$f(0, -8) = (-8)^4 = 4096$$

\therefore Absolute minimum value = -1024 attained at $[-8, 0]$

Absolute maximum value = 4096 attained at $[0, 8]$ and $[0, -8]$

9. Find the coordinates of the point (x, y, z) on the plane $z = 4x + 2y + 4$ which is closest to the origin.

重点题型 closest to the origin.

$$(4x + 2y + 4)^2 + x^2 + y^2 = S(x, y) \quad \text{利用球面程 } (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \quad [a, b, c] \text{ is origin.}$$

$$\frac{\partial S}{\partial y} = 2(4x + 2y + 4)2 + 2y = 4(4x + 2y + 4) + 2y$$

$$\frac{\partial S}{\partial x} = 2(4x + 2y + 4)4 + 2x = 8(4x + 2y + 4) + 2x$$

Find Critical point:

$$\frac{\partial S}{\partial y} = 0 = 4(4x + 2y + 4) + 2y \quad (1)$$

$$\frac{\partial S}{\partial x} = 0 = 8(4x + 2y + 4) + 2x \quad (2)$$

$$(1) \times 2 - (2) =$$

$$8(4x + 2y + 4) + 4y - 8(4x + 2y + 4) - 2x = 0$$

$$4y - 2x = 0$$

$$y = \frac{1}{2}x \quad (3)$$

$$\begin{aligned} \text{Plug (4) into (3)} \\ y = \frac{1}{2}x + \frac{16}{21} = -\frac{16}{42} \end{aligned}$$

$$\text{and } \because z = 4x + 2y + 4$$

$$\begin{aligned} \therefore z &= 4x(-\frac{16}{42}) + 2x(-\frac{16}{42}) + 4 \\ &= 0.19 \end{aligned}$$

$$\text{Plug (3) into (1)} \\ 4(4x + 2y + 4) + x = 0$$

$$16x + 8x + 16 + x = 0$$

$$21x = -16$$

$$x = -\frac{16}{21} \quad (4)$$

$$\therefore [-\frac{16}{21}, -\frac{16}{42}, 0.19]$$



1. Find the maximum and minimum values of $f(x, y) = 5x + y$ on the ellipse $x^2 + 4y^2 = 1$

Method of Lagrange multipliers

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$
(Assuming that these extreme values exist and $\nabla g \neq 0$ on the surface $g(x, y, z) = k$)

(a) Find all values of λ , y , z and ∇ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step(a). The largest of these values is the maximum value of f ; the smallest is the minimum value of f .

$$\text{Let } f = 5x + y \quad g = x^2 + 4y^2 - 1$$

$$\nabla f = 5\mathbf{i} + \mathbf{j} \quad \nabla g = 2x\mathbf{i} + 8y\mathbf{j}$$

$$\text{Let } \nabla f = \lambda \nabla g$$

$$\therefore [5, 1] = \lambda [2x, 8y]$$

$$2x\lambda = 5 \Rightarrow x = \frac{5}{2\lambda}$$

$$8y\lambda = 1 \Rightarrow y = \frac{1}{8\lambda}$$

$$\text{and } x^2 + 4y^2 = 1$$

$$\therefore \left(\frac{5}{2\lambda}\right)^2 + 4\left(\frac{1}{8\lambda}\right)^2 = 1$$

$$\Rightarrow \lambda^2 = \frac{101}{16}$$

$$\therefore \lambda = \pm \sqrt{\frac{101}{16}} = \pm \frac{1}{4}\sqrt{101}$$

$$\text{When } \lambda = \pm \frac{1}{4}\sqrt{101}$$

$$\therefore x = \frac{5}{2\lambda} = \pm \frac{10}{\sqrt{101}}$$

$$y = \pm \frac{1}{2\sqrt{101}}$$

2. Find the maximum and minimum values of $f(x, y, z) = 5x + 1y + 3z$ on the sphere

$$x^2 + y^2 + z^2 = 1$$

maximum value =

minimum value =

Let $f = 5x + 1y + 3z \quad g = x^2 + y^2 + z^2 - 1$

$$\nabla f = 5\mathbf{i} + 1\mathbf{j} + 3\mathbf{k} \quad \nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

Let $\nabla f = \lambda \nabla g$

$$\Rightarrow [5, 1, 3] = \lambda [2x, 2y, 2z]$$

$$\begin{aligned} 2x\lambda &= 5 \Rightarrow x = \frac{5}{2\lambda} \\ 2y\lambda &= 1 \Rightarrow y = \frac{1}{2\lambda} \\ 2z\lambda &= 3 \Rightarrow z = \frac{3}{2\lambda} \end{aligned} \quad \left. \right\} (1)$$

and $x^2 + y^2 + z^2 = 1$

$$\therefore \left(\frac{5}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = 1 \Rightarrow \lambda = \pm \frac{1}{2\sqrt{35}}$$

Plug $\lambda_1 = \pm \frac{1}{2\sqrt{35}}$ into (1) to get x_1, y_1, z_1 .

Plug $\lambda_2 = -\frac{1}{2\sqrt{35}}$ into (1) to get x_2, y_2, z_2

Using x_1, y_1, z_1 to get max value = 5.9

Using x_2, y_2, z_2 to get min value = -5.9.

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Find the maximum and minimum volumes of a rectangular box whose surface area equals 400 square cm and whose edge length (sum of lengths of all edges) is 320 cm. Hint: It can be deduced that the box is not a cube, so if x , y , and z are the lengths of the sides, you may want to let x represent a side with $x \neq y$ and $x \neq z$.

Max value is: occurring at:

Min value is: occurring at:

$$\begin{cases} 4(x+y+z) = 320 \\ 2(xy+yz+xz) = 4000 \\ V = xyz \end{cases}$$

$$\nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

$$\nabla g = (2y+2z)\mathbf{i} + (2x+2z)\mathbf{j} + (2x+2y)\mathbf{k}$$

$$\nabla h = 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$$

$$\Rightarrow \begin{cases} yz = (2y+2z)\lambda + 4\mu \\ xz = (2x+2z)\lambda + 4\mu \\ xy = (2x+2y)\lambda + 4\mu \\ 2xy + 2xz + 2yz = 4000 \\ 4x + 4y + 4z = 320 \end{cases} \Rightarrow \begin{array}{l} x_1 = 40 \quad y_1 = 20 \quad z_1 = 20 \\ x_2 = \frac{40}{3} \quad y_2 = \frac{100}{3} \quad z_2 = \frac{100}{3} \end{array}$$

$$\therefore V_{\max} = xyz = 40 \times 20 \times 20 = 16000$$

$$V_{\min} = xyz = \frac{40}{3} \times \frac{100}{3} \times \frac{100}{3} = \frac{400000}{27}$$



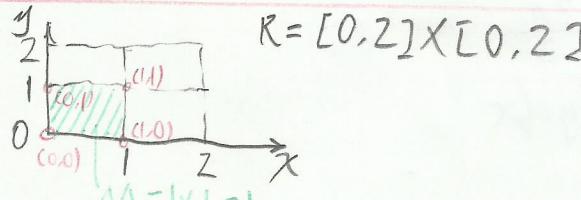
- (c) Consider the solid that lies above the square (in the xy -plane) $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 64 - x^2 - 2y^2$

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- (A) Estimate the volume by dividing R into 4 equal squares and choosing the sample points to lie in the lower left hand corners.

P666 print out

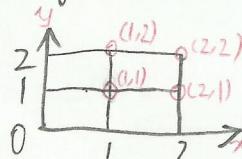
$$V \approx \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$$



Left hand: $\Delta A = 1 \times 1 = 1$

$$\begin{aligned} V &\approx f(0,0)\Delta A + f(0,1)\Delta A + f(1,0)\Delta A + f(1,1)\Delta A \\ &= 64 \times 1 + 62 \times 1 + 63 \times 1 + 61 \times 1 \\ &= 250 \end{aligned}$$

- (B) Estimate the volume by dividing R into 4 equal squares and choosing the sample points to lie in the upper right hand corners.



$$\begin{aligned} V &\approx f(1,1)\Delta A + f(1,2)\Delta A + f(2,1)\Delta A + f(2,2)\Delta A \\ &= 61 + 55 + 58 + 52 \\ &= 226. \end{aligned}$$

- (C) What is the average of the two answers from (A) and (B)

$$A = (250 + 226)/2 = 238$$

2. Using geometry, calculate the volume of the solid under $z = \sqrt{1-x^2-y^2}$ and over the circular disk $x^2+y^2 \leq 1$.

For $z = \sqrt{1-x^2-y^2} \Rightarrow z^2 = 1 - x^2 - y^2 \Rightarrow x^2 + y^2 + z^2 = 1 \quad (z \geq 0)$

$$V = \left(\frac{4}{3}\pi r^3\right)/2 = \left(\frac{4}{3}\pi 1^3\right)/2 = \frac{4\pi}{3} \times \frac{1}{2} = \frac{2\pi}{3}$$

4. Evaluate the iterated integral $\int_1^2 \int_1^2 (2x+y)^{-2} dy dx$.

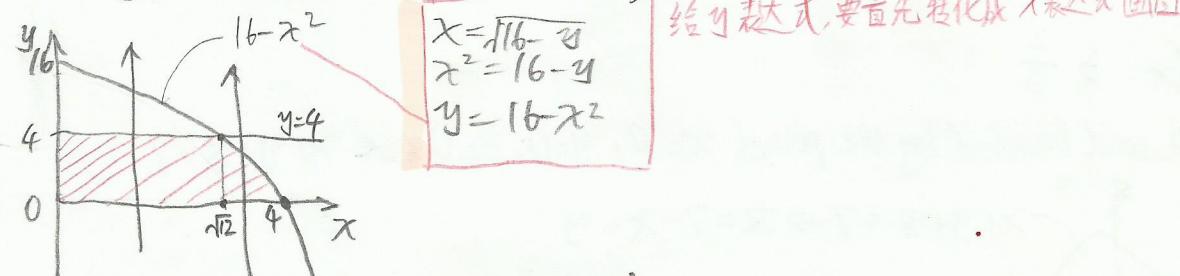
$$\begin{aligned} &= \int_1^2 \int_1^2 (2x+y)^{-2} d(2x+y) dx \\ &= \int_1^2 \left[\frac{(2x+y)^{-1}}{-1} \right]_1^2 dx \\ &= \int_1^2 \left[-\frac{1}{(2x+y)} \right]_1^2 dx \\ &= \int_1^2 \left[-\frac{1}{2x+2} + \frac{1}{2x+1} \right] dx \\ &= -\frac{1}{2} \int_1^2 \frac{1}{2x+2} d(2x+2) + \frac{1}{2} \int_1^2 \frac{1}{2x+1} d(2x+1) \\ &= -\frac{1}{2} [\ln(2x+2)]_1^2 + \frac{1}{2} [\ln(2x+1)]_1^2 \\ &= -\frac{1}{2} [\ln(6) - \ln(4)] + \frac{1}{2} [\ln(5) - \ln(3)] \\ &\approx 0.05268 \end{aligned}$$

key $\int \frac{1}{x} dx = \ln x$

Type 2

7. Consider the integral $\int_0^4 \int_{\sqrt{16-y}}^{4} f(x,y) dx dy$. If we change the order of integration we obtain the sum of two integrals:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) dy dx + \int_c^d \int_{g_3(x)}^{g_4(x)} f(x,y) dy dx$$



$$\therefore \int_0^{\sqrt{12}} \int_0^4 f(x,y) dy dx + \int_{\sqrt{12}}^4 \int_0^{16-x^2} f(x,y) dy dx$$

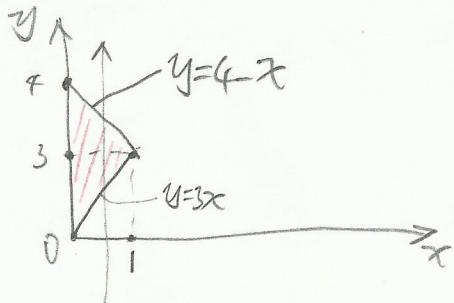
9. In evaluating a double integral over a region D , a sum of iterated integrals was obtained as follows:

$$\iint_D f(x,y) dA = \int_0^3 \int_{\frac{1}{3}y}^{4-y} f(x,y) dx dy + \int_3^4 \int_0^{4-y} f(x,y) dx dy.$$

Sketch the region D and express the double integral as an integral with reversed order of integration.

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) dy dx.$$

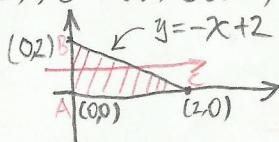
$$x = \frac{1}{3}y \Rightarrow y = 3x \quad x = 4 - y \Rightarrow y = 4 - x$$



$$\therefore \int_0^1 \int_{3x}^{4-x} f(x,y) dy dx$$

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3. Evaluate the double integral $I = \iint_D xy \, dA$ where D is the triangular region with vertices $(0,0), (2,0), (0,2)$

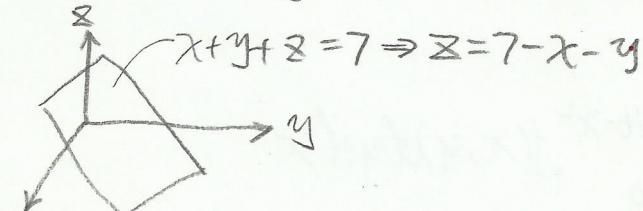
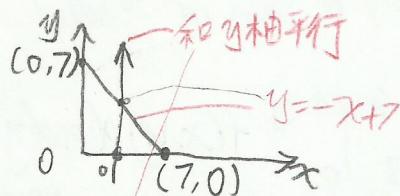


use Type 2.

$$\int_0^2 \int_0^{-x+2} xy \, dy \, dx = \frac{2}{3}$$

4. Find the volume of the solid bounded by the planes $x=0, y=0, z=0$ and $x+y+z=7$.

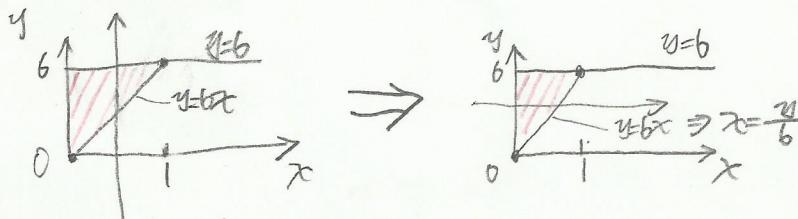
key!

when $z=0$ 

$$\int_0^7 \int_0^{-x+7} 7-x-y \, dy \, dx = \frac{343}{6}$$

5. Consider the integral $\int_0^1 \int_{y^2}^6 f(x,y) \, dy \, dx$, sketch the region of integration and change the order of integration.

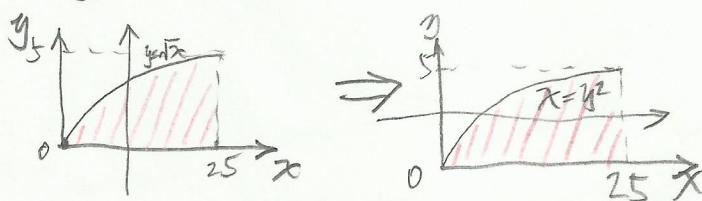
$$\int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y) \, dx \, dy$$



$$\therefore \int_0^6 \int_0^{\sqrt{y}} f(x,y) \, dx \, dy$$

6. Consider the integral $\int_0^5 \int_0^{25} f(x,y) \, dy \, dx$, sketch the region of integration and change the order of integration.

$$\int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y) \, dx \, dy$$



$$\therefore \int_0^5 \int_{y^2}^{25} f(x,y) \, dx \, dy$$

Section 12.3

1. Using polar coordinates, evaluate the integral $\iint_R \sin(x^2 + y^2) dA$ where R is region $x^2 + y^2 \leq 64$.

Change to polar coordinates in a double integral.

If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b$, $a \leq \theta \leq b$, where $0 \leq \beta - a \leq 2\pi$, then

$$\iint_R f(x, y) dA = \int_a^b \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta$$

$$\therefore 1 \leq x^2 + y^2 \leq 64$$

$$x^2 + y^2 = r^2$$

$$\therefore 1 \leq r \leq 8 \quad r \text{ should} = 0$$

$$\therefore \int_0^{2\pi} \int_1^8 \sin(r^2) r dr d\theta$$

$$= \int_0^{2\pi} \int_1^8 \sin(r^2) dr^2 d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} [-\cos(r^2)]_1^8 d\theta$$

$$= \frac{1}{2} [-\cos(64) - (-\cos(1))] \cdot 2\pi$$

$$= [\cos(1) - \cos(64)] \cdot \pi$$

3. Find the volume of the ellipsoid $x^2 + y^2 + 7z^2 = 100$

$$x^2 + y^2 - 100 = -7z^2$$

$$z = \sqrt{\frac{x^2 + y^2 - 100}{-7}}$$

$$\text{When } z=0 \Rightarrow x^2 + y^2 = 100$$

$$y^2 = 100$$

$$r = 10$$

$$\therefore V = 2 \int_0^{2\pi} \int_0^{10} \sqrt{\frac{y^2 - 100}{-7}} r dr d\theta$$

$$= 2 \times \frac{2000}{21} \sqrt{7} \pi$$

$$= \frac{4000}{21} \sqrt{7} \pi$$

4. Find the volume of the solid enclosed by the paraboloids $z = 16(x^2 + y^2)$ and $z = 50 - 16(x^2 + y^2)$

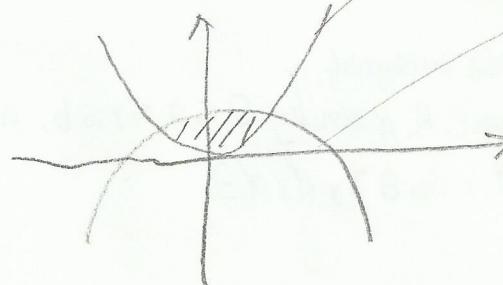
$$\text{Let } r^2 = x^2 + y^2$$

$$\therefore z = 16r^2 = 50 - 16r^2$$

$$\therefore 32r^2 = 50$$

$$r^2 = \frac{50}{32}$$

$$r = \sqrt{\frac{50}{32}}$$



$$V_1 = \int_0^{2\pi} \int_0^r 16r^2 r dr d\theta$$

$$= 4\left(\frac{50}{32}\right)^2 2\pi$$

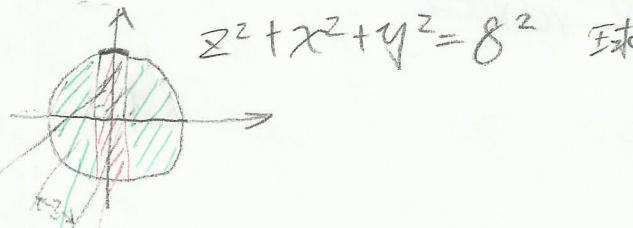
$$\int_0^{2\pi} \int_0^r 50 - 16r^2 r dr d\theta$$

$$V_2 = \frac{12 \cdot 50^2}{32^2} 2\pi$$

$$V_2 - V_1 = \frac{12 \cdot 50^2}{32^2} 2\pi - \frac{4 \cdot 50}{32^2} 2\pi$$

$$= \frac{8 \cdot 50^2}{32^2} 2\pi$$

5. A cylindrical drill with radius 3 is used to bore a hole thought the center of radius 8. Find the volume of the ring shaped solid that remains.



$$\therefore z = \sqrt{8^2 - x^2 - y^2}$$

$$V = 2 \iint \sqrt{8^2 - x^2 - y^2} dx dy$$

$$= 2 \int_0^{2\pi} \int_0^3 \sqrt{8^2 - r^2} r dr d\theta$$

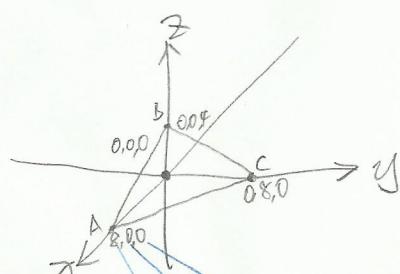
$$= \frac{1024}{3} \pi - \frac{110}{3} \sqrt{55} \pi$$

$$\therefore V_{\text{remains}} = \frac{4}{3} \pi R^3 - 2V = \frac{4}{3} \pi 8^3 - 2\left(\frac{1024}{3} \pi - \frac{110}{3} \sqrt{55} \pi\right)$$

$$= \frac{220}{3} \sqrt{55} \pi$$

Review for Mat 267

Sec 12.5 P2 **key**
 Evaluate the triple integral $\iiint_E xy \, dV$ where E is the solid tetrahedron with vectors
 $[0,0,0], [8,0,0], [0,8,0], [0,0,4]$. 题型：三重积分四面体定上限



$$\vec{AB} = [-8, 0, 4]$$

$$\vec{AC} = [-8, 8, 0]$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -8 & 0 & 4 \\ -8 & 8 & 0 \end{vmatrix} = 32i + (-32)j + 64k = 32i + 32j + 64k$$

Find Plane ABC:

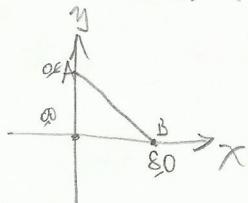
$$32(x-8) + 32(y-0) + 64(z-0) = 0$$

$$(x-8) + (y-0) + 2(z-0) = 0$$

$$x-8 + y + 2z = 0$$

$$\therefore z = \frac{8-x-y}{2}$$

Find 投影在 xy 平面上的方程.

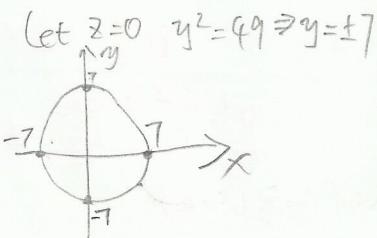
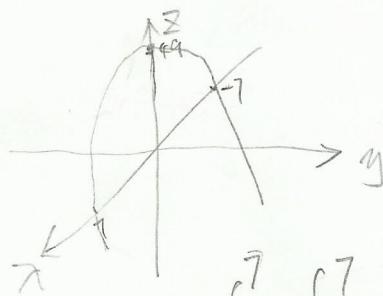


$$\Rightarrow \text{line AB: } \frac{y-0}{x-8} = \frac{8-0}{0-8} \Rightarrow \frac{y}{x-8} = \frac{8}{-8} \Rightarrow -8y = 8x - 64 \Rightarrow y = -x + 8$$

$$\begin{aligned} \therefore \iiint_E xy \, dV &= \int_0^8 \int_0^{x+8} \int_0^{\frac{8-x-y}{2}} xy \, dz \, dy \, dx \\ &= \int_0^8 \int_0^{x+8} \left[xyz \right]_0^{\frac{8-x-y}{2}} dy \, dx \\ &= \int_0^8 \int_0^{x+8} xy \cdot \frac{8-x-y}{2} dy \, dx \\ &= \frac{2048}{15} \end{aligned}$$

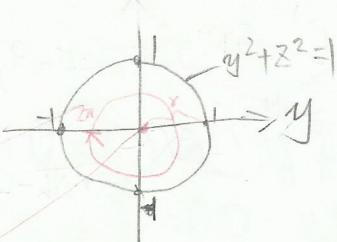
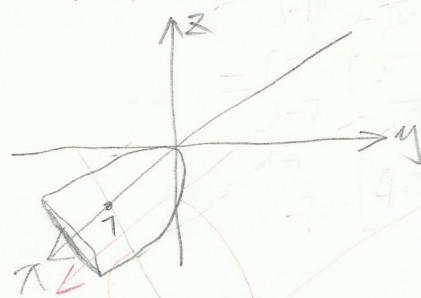
P3. Evaluate the triple integral $\iiint_E x^8 e^y dV$ where E is bounded by the parabolic cylinder $z=49-y^2$ and the planes $z=0$, $z=7$, and $x=-7$.

投物面, X, y 投影是圆.



$$\int_{-7}^7 \int_{-7}^7 \int_0^{49-y^2} x^8 e^y dz dy dx$$

P4. Evaluate the triple integral $\iiint_E x dV$ where E is the solid bounded by the paraboloid $x=7y^2+7z^2$ and $x=7$.



$$\int_{-1}^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{7y^2+7z^2}^7 x dz dy dx = \frac{49}{3}\pi$$

Let $x=0$
 $0=7y^2+7z^2$
 $-7y^2=7z^2$
 $y^2=-z^2$
 $y=\pm\sqrt{-z^2}$

$$V = \int_0^{2\pi} \int_0^1 \int_{7r^2}^7 x dr d\theta$$

计算方法: key!
 $\text{at: } \iiint f(x, \theta, z) dV = \iiint f(r, \theta, z) r^2 r dr d\theta$

$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$

$$= \int_0^{2\pi} \int_0^1 \left[\frac{x^2}{2} \right]_{7r^2}^7 r dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left(\frac{49}{2} - \frac{49r^4}{2} \right) r dr d\theta$$

$$= \frac{49}{2} \int_0^{2\pi} \int_0^1 (1-r^4) r dr d\theta$$

$$= \frac{49}{4} \int_0^{2\pi} \int_0^1 (1-r^4) dr^2 d\theta$$

$$(r^4)^2$$

$$= \frac{49}{4} \int_0^{2\pi} \left[r^2 - \frac{r^5}{5} \right]_0^1 d\theta$$

$$= \frac{49}{4} \int_0^{2\pi} \frac{2}{3} d\theta$$

$$= \frac{49}{4} \times \frac{2}{3} [\theta]_0^{2\pi}$$

$$= \frac{49}{6} \times 2\pi$$

$$= \frac{49}{3}\pi$$

Sec 13.4 P1

Let C be the positively oriented circle $x^2+y^2=1$. Use Green's Theorem to evaluate the line integral: $\int_C 18ydx+6xdy$

Green Theorem:

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C . If P and Q have continuous partial derivatives on an open region that contains D , then

$$\int_C Pdx+Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad dA \text{ 可以是 } dx dy \text{ 或 } r dr d\theta$$

Solution:

$$P=18y \sim \frac{\partial P}{\partial y} = 18$$

$$Q=6x \sim \frac{\partial Q}{\partial x} = 6$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 6-18=-12$$

$$\therefore \int_C 18ydx+6xdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \iint_D -12 dx dy$$

$$\because x^2+y^2=1 \Rightarrow r^2=1 \Rightarrow r=1$$

$$\begin{aligned} \therefore \int_C 18ydx+6xdy &= \int_0^{2\pi} \int_0^1 -12r dr d\theta = -12 \times \int_0^{2\pi} \left[\frac{r^2}{2} \right]_0^1 d\theta \\ &= -12 \int_0^{2\pi} \frac{1}{2} d\theta \\ &= -6 \times 2\pi = -12\pi \end{aligned}$$

P₂.

Let C be the positively oriented square with vertices (0,0), (1,0), (1,1) (0,1). Use Green's Theorem to evaluate the line integral

$$\int_C 10y^2x \, dx + 2x^2y \, dy$$

$$P = 10y^2x \sim \frac{\partial P}{\partial y} = 10x \times 2y = 20xy$$

$$Q = 2x^2y \sim \frac{\partial Q}{\partial x} = 2y \times 2x = 4xy$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4y - 20xy = -16xy$$

$$\begin{aligned}\therefore \int_C 10y^2x \, dx + 2x^2y \, dy &= \int_0^1 \int_0^1 -16xy \, dx \, dy \\ &= \int_0^1 \left[-16y \frac{x^2}{2} \right]_0^1 \, dy \\ &= \int_0^1 -8y \, dy \\ &= \left[-8 \frac{y^2}{2} \right]_0^1 \\ &= -4\end{aligned}$$

